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ON G SPACE THEORY

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This paper presents an examinations of the G space theory that was established recently for a unified formulation of compatible and incompatible models for mechanics problems using the finite element and meshfree settings. Using the generalized gradient smoothing technique, we first give a general definition for G spaces with more details on the G^1 space containing both continuous and discontinuous functions. The physical meanings and implications of various numerical treatments used in the G space theory are discussed in detail. Both normed and un-normed G spaces are discussed with emphases on the normed G spaces. Some important properties and a set of useful inequalities for the normed G spaces are proven in theory and analyzed in detail. Because discontinuous functions are allowed in a G space, much more types of function approximation methods and techniques can be used to create shape functions for numerical models. These models can be compatible and incompatable but are all stable and converge to the exact solution to the corresponding strong formulation as long as it is well-posed, based on the normed G space theory. Methods based on normed G space theory does not use the derivatives of the displacement functions in the formulation and is known as the weakened weak (W^2) formulation that has a number of attractive properties such as conformability, softness, upper/lower bound, superconvergence, and ultra accuracy.

Keywords: Numerical method; meshfree method; G space; weakened weakform; point interpolation method; finite element method; solution bound; variational principle; generalized smoothed Galerkin; compatibility.

1. Introduction

To solve applied mechanics problems, many powerful numerical methods based on the standard weakform formulation have been developed in the past half a century, based on the well-known and well-established Hilbert space theory. The finite element method (FEM) [Zienkiewicz and Taylor (2000); Liu and Quek (2003); Oliveira Eduardo and De Arantes (1968)] is a typical such a weakform method and the meshfree methods [see, e.g. Liu and Liu (2003); Liu and Gu (2005); Liu (2009)] are important additions. The FEM is very thoroughly developed and is currently

the most widely used reliable numerical tool with many commercial software packages available. However, there are some issues related to the FEM. For example, the "overly-stiff" phenomenon of a fully-compatible FEM model of assumed displacement based on the Galerkin weakform. Such an "overly-stiff" behavior is often observed as the so-called "locking" phenomenon, and it is also responsible for the inaccuracy in stress solutions for triangular mesh, and mesh distortion related problems. The analysts often prefer using linear triangular types of mesh as they can be created much more easily and even automatically, and applicable to much more wide class of problems with lower regularity in solution. However, the FEM does not like such elements and often gives very poor accuracy. In some commercial software packages, analysts often get a warning when opting for triangular elements precisely for this kinds of reasons.

Many efforts have been made in resolving these issues, especially in the area of hybrid FEM formulations (see, e.g. [Pian and Wu (2006)]). It is, however, the author's opinion that the root of all the above-mentioned problem is the way that the semi-norm is evaluated in the Hilbert space theory, and hence a *fundamental* change in function space theory is necessary, after working with the Hilbert space for as much as half a century. Recently, a smoothed FEM (or SFEM) [Liu, Dai and Nguyen (2007); Liu *et al.* (2007); Dai, Liu and Nguyen (2007)] has also been formulated by combining the FEM procedures and the *gradient smoothing operation* known as distributional (weak) derivatives in classic sense. The smoothing operation has been used in the nonlocal continuum mechanics [Eringen and Edelen (1972); Zhang, Liu and Han (2006)], and the smoothed particle hydrodynamics (SPH) [Liu and Liu (2003); Lucy (1977); Liu, Liu and Zong (2008); Monaghan (1982)]. It was used in resolving the material instabilities [Chen, Wu and Belytschko (2000)] and spatial instability in nodal integrated meshfree methods [Chen *et al.* (2001)], and recently in obtaining upper bound solutions in meshfree point interpolation methods [Liu *et al.* (2005); Zhang *et al.* (2007)]. In the SFEM, the smoothing operations were performed over smoothing domains created associated with cells obtained by subdivision of elements of higher orders. The SFEM works very effectively for solid mechanics problems and n-sided polygonal elements and very heavily distorted mesh can be used [Dai, Liu and Nguyen (2007)]. Detailed theoretical aspects including stability and convergence about SFEM can be found in [Liu *et al.* (2007)]. The study of SFEM has also clearly shown that the smoothing operation on strains alters the assumed strain field in a proper fashion using the cell-based smoothing domains to ensure the stability and the convergence, and ultimately gives the SFEM some very good features. A more general cell-based smoothed model that works for triangular cells/elements is the recent cell-based smoothed point interpolation method (CS-PIM or CS-RPIM) [Liu and Zhang (2009)], using meshfree PIM or RPIM shape functions.

In the development of meshfree methods, the node-based strain smoothing [Chen *et al.* (2001)] was extended to a generalized strain smoothing technique [Liu (2008a)] that is the foundation of the node-based smoothed point interpolation method (NS-PIM), known also as originally the LC-PIM [Liu *et al.* (2005);

Zhang *et al.* (2007)]. The NS-PIM is formulated using PIM [Liu and Gu (2001; 2005)] or RPIM [Wang and Liu (2002)] shape functions of Kronecker delta for easy treatments of essential boundary conditions. Although these nodal PIM and RPIM shape functions are generally discontinuous, the NS-PIM or (NS-RPIM [Liu and Li *et al.* 2006]) was found at least linearly conforming, can produce much better stress solution, much more tolerant to mesh distortion, works very well for triangular elements, and more importantly it can often provide upper bound solution in energy norm. A node-based SFEM (or NS-FEM) has also been formulated within the framework of FEM settings. The NS-FEM can be viewed as a special case of NS-PIM, but based on n-sided polygonal element mesh [Liu *et al.* (2009)], and has quite similar properties as NS-PIM. It was found that NS-PIM and NS-FEM behaves "overly-soft" leading to the so-called *temporal* instability when used to solve dynamic problems. To reduce the soft effects, a very effective edge-based smoothed FEM (ES-FEM) for 2D problems [Liu *et al.* (2008)] and face-based smoothed FEM (FS-FEM) for 3D problems [Nguyen *et al.* (2008)] have been recently formulated. The ES-FEM (or FS-FEM) not only produces very accurate solution, but also is temporally stable and no spurious modes, and hence works very well for dynamic problems. The linear ES-FEM using triangular mesh has been found "ultra-accurate" model with "close-to-exact" stiffness. A more general edge-based smoothed model is the recent ES-PIM [Liu and Zhang (2008a)]. The formulation of all these methods are termed as weakened weak (W^2) formulation [Liu (2008b)] that have a common theoretical base on the G space theory [Liu (2008); Liu and Zhang (2009)].

The first initial conceptual presentation on the G space was originated in [Liu (2008a)] without very precise definition on norms. The more precise definition of G^1 space with norms for W^2 formulations was then given in [Liu and Zhang (2009)], and proofs and more detailed discussion on some important definitions, inequalities, properties as well as applications were provided recently in [Liu (2008b)]. It is clear that the G^1 space is a much more accommodative space than the H^1 space, and forms the foundation of a much more general class of W^2 methods including compatible and incompatible methods with FEM and meshfree settings. Many W^2 models have been found with number of attractive properties such as conformability, softness, upper/lower bound, superconvergence, ultra accuracy, etc, and they all work well with triangular cells.

In this work, we conduct a detailed analysis on the G spaces, normed or unnormed, that host discontinuous displacement functions with the generalized gradient smoothing technique for semi-norm definition [Liu (2008b)]. We discuss about the physical meanings and numerical implications for a number of important numerical treatments, and then provide important properties of the functions in a G space, including a set of key inequalities that are the foundation for the stability and the convergence of a numerical method seeking solutions in a proper G space. These properties ensure that a numerical method developed using functions in the normed G space will be always spatially stable and converges to exact solutions, as long as the original mechanics problem is well-posed.

2. Function Approximation

2.1. *Domain discretization by triangulation*

Consider a d-dimensional problem domain of $\Omega \in \mathbb{R}^d$ bounded by Γ that is "*Lipschitzian*". We speak "open" domain that does not include the boundary of the domain. When we refer to a "closed" domain we will specifically use a box: $\boxed{\Omega} = \Omega \cup \Gamma$. The problem domain Ω is divided into cells (in meshfree settings) or elements (in FEM settings) of generally polygons of multiple sides [Dai, Liu and Nguyen (2007); Liu *et al.* (2009)]. Although we do not exclude other shape of cells/elements, we use by default triangular types of cells/elements for easy generation for complicated domains using the well-established triangulation algorithms such as the well known Delaunay algorithms. For 1D problems a cell is defined in $\mathbb{R}^1$ and is simply a line segment, for 2D problems it is defined in $\mathbb{R}^2$ and becomes a triangle, and for 3D problems it is defined in $\mathbb{R}^3$ and is tetrahedron. The domain is divided with N_e cells/elements that are connected at N_n nodes. The "length" of the cell is denoted *generally* as h and it can be different from cell to cell. For uniform discretization and h becomes the characteristic dimension of cells. In the case of non-uniform discretization, we let

$$h_{\max} = \max_{q=1,\ldots,N_e}(h_q), \quad h_{\min} = \min_{q=1,\ldots,N_e}(h_q), \tag{1}$$

and we should assume that the ratio of the smallest and largest cell dimensions is bounded:

$$h_{\max}/h_{\min} = c_{rh} < \infty. \tag{2}$$

In such cases, the largest $h_{\max}$ become the characteristic dimension of the cells. The characteristic dimension of cells can also be defined as some kind of averaged cell length, such as

$$h = \sqrt{4A_e/\sqrt{3}}, \tag{3}$$

where A_e is the "average" area the triangular cells defined as $A_e = A/N_e$ in which A is the area of the entire problem domain. In Eq. (3), we assume the cells are all isolateral triangles. Alternatively, we can simply use

$$h = \sqrt{2A_e}, \tag{4}$$

where we assume the cells are all right-isosceles triangles. Note that Eqs. (1), (3) and (4) are "equivalent" in the sense that "controlling h defined in any of these three ways can put the entire mesh under control". We require the ratio c_{rh} to be bounded, and also the inner angles θ of the triangles should be strictly larger than zero and less than 180 degree in theory. In practical models, we often require $15 < \theta < 120$, which can be achieved using the available algorithms. Note our division is also seamless: $\boxed{\Omega} = \bigcup_{i=1}^{N_e} \boxed{\Omega}_i^e$, where the box stands for closed domains. Such a triangulation is *generally* denoted as T_h: a collection of all Ω_i^e $(i = 1, 2, \ldots, N_e)$.

2.2. *Basis and displacement approximation*

In establishing any discrete numerical model, field functions have to be approximated over the problem domain using a set of nodal values of the functions and so-called *basis*. Given a linear space S of dimension N_n, a set of N_n members of functions $\phi_n \in S, n = 1, 2, \ldots, N_n$ is a basis for S if and only if $\forall w \in S, \exists$ unique $\alpha_n \in \mathbb{R}$ such that

$$w = \sum_{n=1}^{N_n} \alpha_n \phi_n. \tag{5}$$

Function ϕ_n in the *basis* is often given in the form of *nodal* shape functions, and hence the basis is also termed as *nodal basis* in the context of FEM and meshfree methods. Equation (5) implies that the nodal shape functions must be *linearly independent* to be qualified to form a basis. In the FEM these linearly independent shape functions are created based on elements using mostly polynomial basis functions, and the linearly independence is ensured by element topology and properly controlled coordinate mapping (see, e.g. [Liu and Quek (2003)]). In the meshfree methods it is based on local nodes using both polynomial and radial basis functions [Liu and Gu (2005)], generally no mapping is needed, and the linearly independence is ensured by the use of proper basis functions and/or proper local nodes selection with the help of a background cells [Liu (2009)]. The often used meshfree shape functions are PIM, RPIM, least square, moving least square, and SPH shape functions. The procedure for the creation of these types of shape functions are rather standard and can be found in great detail in textbooks [Liu and Gu (2005); Liu (2009)], and hence are omitted here. In this paper we assume that there are always ways to obtain a set of independent nodal shape functions for given a set of nodes with a set of elements or background cells in the problem domain Ω. For the convenience of discussion, we require the shape functions ϕ_n to have the following properties. (1) partitions of unity:

$$\sum_{n=1}^{N_n} \phi_n(\mathbf{x}) = 1, \tag{6}$$

that ensures correct representation of rigid body movement every where in the problem domain, and (2) the Kronecker delta property:

$$\phi_i(\mathbf{x}_j) = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}, \tag{7}$$

which allows easy treatments for essential/Dirichlet boundary conditions [Liu and Gu (2005); Liu (2009)]. Once the nodal shape functions are obtained, an assumed (displacement component) function can be given as

$$u^h(\mathbf{x}) = \sum_{n \in S_s} \phi_n(\mathbf{x}) d_n, \tag{8}$$

where $\mathbf{x} = \{x_1, x_2\}^T$, S_s is the set of the nodes in the local support domain containing $\mathbf{x}$, d_n is the nodal value of a component of the assumed displacement function.

Finally, for consistence reasons, we require the nodal shape functions being *polynomial* linearly complete, at least when h approaches zero. The linear completion can be simply achieved by adding in polynomials with constant and linear terms [Liu (2009)], so that the resultant models can pass the standard patch tests. When the pure RBFs are used in constructing the RPIM shape functions, the *polynomial* linearly completion is lost. However, such a linear completion is achieved at the limit of h approaching zero [Liu and Gu (2005)]. In these cases, the model may not be able to pass the standard patch tests for a finite discrete model, but will converge as h approaching zero [Liu (2009)], if the numerical model is stable.

2.3. *Point interpolation*

The definition of the interplant is more general in G space theory, and is given as follows. Given any function $w \in S$ where S is a linear space, the *interpolant* $\mathcal{I}_h w$ creates a function that lives in a subspace $S_h \subset S$ with N_n dimensions: $\mathcal{I}_h w \in S_h$ where

$$\mathcal{I}_h w(x) = \sum_{n=1}^{N_n} w(x_n)\phi_n(x). \tag{9}$$

With the Delta function property given in Eq. (7), we shall have

$$\mathcal{I}_h w(x_n) = w(x_n), \quad n = 1, 2, \ldots, N_n. \tag{10}$$

All the FEM shape functions and the more general PIM and RPIM shape functions can be used for such an interpolation. Note the interpolant generated by the above *point interpolation* does not necessarily live in an H^1 space, which is important in the Hilbert space theory in functional analysis. However, it lives in a G^1 space to be defined later. Therefore, the point interpolation defined here is generally different from those defined in the FEM settings where the interpolation is element-based with proper mapping to ensure that the interpolant lives in a desired H space. Such an interpolant in an H space lives also in the corresponding G space, because a G space is much more accommodative as will be discussed later.

2.4. *Integral representation of function*

In our definition of G spaces, we need to approximate also the gradient of functions, for which we now introduce the so-called integral representation of a function given by (see, e.g. [Liu (2009)]):

$$\widehat{w}(\mathbf{x}) = \int_{\Omega_{\mathbf{x}}^s} w(\xi)\widehat{W}(\mathbf{x} - \xi)\mathrm{d}\xi, \tag{11}$$

where w is assumed integrable in $\Omega_{\mathbf{x}}^s$ in the sense of *Lebesgue* integration that allows occasional *omissions* at finite points; $\widehat{w}(\mathbf{x})$ is termed as the *smoothed* function over

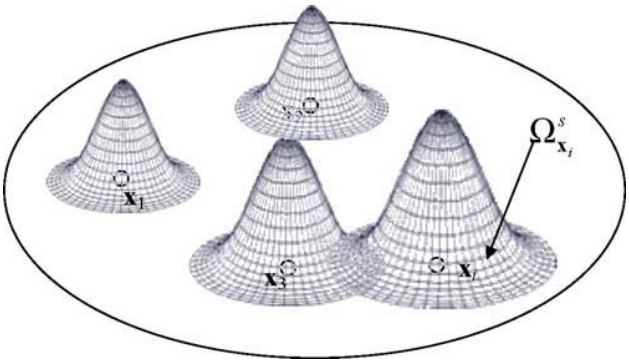


Fig. 1. Smoothing domains for $\Omega^s_{\mathbf{x}}$ integral representation of a function at $\mathbf{x}$, over which the smooth function is defined. The smoothing domain can, in general, be different for different $\mathbf{x}$ and they can also overlap. The smoothing functions can also be different for different $\mathbf{x}$.

$\Omega^s_{\mathbf{x}} \subset \Omega$ to represent w at $\mathbf{x}$ in an integral form, and $\widehat{W}(\mathbf{x}-\xi)$ is a pre-described smoothing function defined in the smoothing domain $\Omega^s_{\mathbf{x}}$ for the point at $\mathbf{x}$, as shown in Fig. 1. Note that in general the smoothing domain is "moving" and thus the domains for different $\mathbf{x}$ can overlap. In this paper we require the smoothing function $\widehat{W}$ satisfying the following basic conditions:

$$\widehat{W}(\mathbf{x}-\xi) \geq 0 \ \forall \xi \in \Omega^s_{\mathbf{x}}, \ \text{(positivity)}, \tag{12}$$

$$\int_{\Omega^s_{\mathbf{x}}} \widehat{W}(\mathbf{x}-\xi)d\xi = 1, \ \text{(unity)}, \tag{13}$$

$$\widehat{W}(\mathbf{x}-\xi_1) \leq \widehat{W}(\mathbf{x}-\xi_2), \forall \xi_1, \xi_2 \in \Omega^s_{\mathbf{x}} \quad \text{and} \quad |\mathbf{x}-\xi_1| \leq |\mathbf{x}-\xi_2| \ \text{(decay)}. \tag{14}$$

The unity condition ensures that the constant value of the function can be exactly represented (see, e.g. [Liu and Liu (2003); Monaghan (1982)]). Conditions (12)–(14) leads to the following remark.

Remark 2.1. *Convergence property*: when $\widehat{W}$ satisfies Eqs. (12)–(14) in the moving smoothing domain $\Omega^s_{\mathbf{x}}$, it approaches to the Delta function when $\Omega^s_{\mathbf{x}} \to 0$, and hence the integral representation of a continuous function will be exact (see, e.g. [Liu and Liu (2003)]).

2.4.1. *Integral representation of gradient of a function*

By the same token, the integral representation can also be performed for the derivatives of a function:

$$\widehat{\frac{\partial w_i}{\partial x_j}}(\mathbf{x}) = \int_{\Omega^s_{\mathbf{x}}} \frac{\partial w_i(\xi)}{\partial x_j} \widehat{W}(\mathbf{x}-\xi)d\xi \,, \tag{15}$$

where $\widehat{\frac{\partial w_i}{\partial x_j}}(\mathbf{x})$ denotes the smoothed derivatives of $w_i(\mathbf{x})$. When both $w_i(\mathbf{x})$ and $\widehat{W}(\mathbf{x}-\xi)$ are *continuous* and differentiable in the *closure* of $\Omega_{\mathbf{x}}^s$, the integration by parts can be applied and Eq. (15) becomes:

$$\widehat{\frac{\partial w_i}{\partial x_j}}(\mathbf{x}) = \int_{\Gamma_{\mathbf{x}}^s} w_i(\xi) n_j \widehat{W}(\mathbf{x}-\xi) \mathrm{d}\Gamma - \int_{\Omega_{\mathbf{x}}^s} w_i(\xi) \frac{\partial \widehat{W}(\mathbf{x}-\xi)}{\partial x_j} \mathrm{d}\xi, \tag{16}$$

where n_j is the directional cosine of the outwards normal on $\Gamma_{\mathbf{x}}^s$.

Equations (11), (15) and (16) are standard forms of *smoothing operation*. Other forms were used in the nonlocal continuum mechanics [Eringen and Edelen (1972); Zhang, Liu and Han (2006)], the smoothed particle hydrodynamics [Liu and Liu (2003); Lucy (1977); Monaghan (1982)], stabilizing nodal integrated meshfree methods and restoring conformability [Chen *et al.* (2001)], and restoring conformability and obtaining upper bound solution in meshfree point interpolation methods [Liu *et al.* (2005); Liu and Zhang (2008b)].

2.4.2. *Gradient approximation*

When $w_l(\mathbf{x})$ is discontinuous in $\Omega_{\mathbf{x}}^s$, the integration by parts [Wikipedia] is no longer applicable, and hence we *approximate* the gradient of $w_l(\mathbf{x})$ in the form of:

$$\widehat{\frac{\partial w_l}{\partial x_i}}(\mathbf{x}) \approx \int_{\Gamma_{\mathbf{x}}^s} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi) \mathrm{d}\Gamma - \int_{\Omega_{\mathbf{x}}^s} w_l(\xi) \frac{\partial \widehat{W}(\mathbf{x}-\xi)}{\partial x_i} \mathrm{d}\xi. \tag{17}$$

This generalization given in Eq. (17) is not rigorous in theory, but it is fortunately possible to implement because no differentiation upon w is required on the right-hand-side of Eq. (17). It was first performed in [Liu (2008a)] and provides the foundation for the NS-PIM [Liu and Zhang *et al.* (2005)], ES-PIM [Liu and Zhang (2008a)] and CS-PIM [Liu and Zhang (2009)] methods using incompatible nodal shape functions. Observing Eq. (17), we note that the differentiation on the assumed displacement has now been transferred to on the smoothing functions. Therefore, the continuity requirement on the assumed displacement is reduced by one order, leading the so-called weakened weak (W^2) formulations. The generalization given in Eq. (17) is useful and very important for the establishment of the G space theory. We now note

Remark 2.2. Generalized gradient smoothing

The smoothed gradient defined in Eq. (17) is a generalized concept. It is NOT "the smoothed gradient obtained by smoothing the gradient of the function", because such a gradient does not in general exist (even in distributional sense) as the function may not be continuous! Rigorously speaking, the "smoothed gradient" is the outward *flux* of the assumed displacement field through the smoothing

domain boundary $\Gamma^s_{\mathbf{x}}$. Such an *approximation* preserves the overall *equivalence* of the strain-displacement relation over the smoothing domain for solid mechanics problems, which is important to ensure the energy balance in energy principles using the (generalized) "smoothed" strains. In fact, it is the essential reason why the W^2 formulation such as the generalized smoothed Galerkin (or GS-Galerkin) weakform works for assumed functions in a $\mathbb{G}^1_h$ space [Liu (2009)].

2.4.3. *On treatment of the discontinuity*

The understanding of the approximation of Eq. (17), may not be so straightforward to some. Below is a mathematical derivation revealing some of the details. Let's consider a simple (but without the loss of generality) case of a local smoothing domain $\Omega^s_{\mathbf{x}}$ for point $\mathbf{x}$, in which the function w_l is discontinuous along a line $\Gamma_{\mathbf{x}d}$ as shown in Fig. 2. We now divide the smoothing domain into three enclosed subdomains: the subdomain on the left $\Omega^s_{\mathbf{x}L}$ bounded by $\Gamma_{\mathbf{x}Left} = \Gamma^s_{\mathbf{x}L} \cup \Gamma^-_{\mathbf{x}T} \cup \Gamma^-_{\mathbf{x}d} \cup \Gamma^-_{\mathbf{x}B}$, the subdomain on the right $\Omega^s_{\mathbf{x}R}$ bounded by $\Gamma_{\mathbf{x}Right} = \Gamma^s_{\mathbf{x}R} \cup \Gamma^+_{\mathbf{x}T} \cup \Gamma^+_{\mathbf{x}d} \cup \Gamma^+_{\mathbf{x}B}$, and the subdomain $\Omega^s_{\mathbf{x}d}$ enveloping (counter-clock-wise) the discontinuous line $\Gamma_{\mathbf{x}d}$ bounded by $\Gamma^+_{\mathbf{x}d} \cup \Gamma^-_{\mathbf{x}d}$ (Because $\Gamma^+_{\mathbf{x}d}$ and $\Gamma^-_{\mathbf{x}d}$ are of infinite lose, the area of $\Omega^s_{\mathbf{x}d}$ is zero). In such a subdomain division, w_l and $\widehat{W}(\mathbf{x}-\xi)$ are now *continuous* and differentiable in the *closure* of $\Omega^s_{\mathbf{x}L}$ and $\Omega^s_{\mathbf{x}R}$, respectively. Therefore, Eq. (16) can now be used for these two subdomains, and we shall have

$$\begin{aligned}
\widehat{\frac{\partial w_l}{\partial x_j}}(\mathbf{x}) &= \int_{\Omega^s_{\mathbf{x}}} \frac{\partial w_l(\xi)}{\partial x_j} \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\xi \\
&= \int_{\Omega^s_{\mathbf{x}L}} \frac{\partial w_l(\xi)}{\partial x_j} \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\xi + \int_{\Omega^s_{\mathbf{x}R}} \frac{\partial w_l(\xi)}{\partial x_j} \widehat{W}(\mathbf{x}-\xi)d\xi \\
&\quad + \underbrace{\int_{\Omega^s_{\mathbf{x}d}} \frac{\partial w_l(\xi)}{\partial x_j} \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\xi}_{I_3} \\
&= \underbrace{\int_{\Gamma_{\mathbf{x}Left}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma + \int_{\Gamma_{\mathbf{x}Right}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma}_{I_1} \\
&\quad - \left(\underbrace{\int_{\Omega^s_{\mathbf{x}L}} w_l(\xi) \frac{\partial \widehat{W}(\mathbf{x}-\xi)}{\partial x_i}\mathrm{d}\xi + \int_{\Omega^s_{\mathbf{x}R}} w_l(\xi) \frac{\partial \widehat{W}(\mathbf{x}-\xi)}{\partial x_i}\mathrm{d}\xi}_{I_2} \right) + I_3. \qquad (18)
\end{aligned}$$

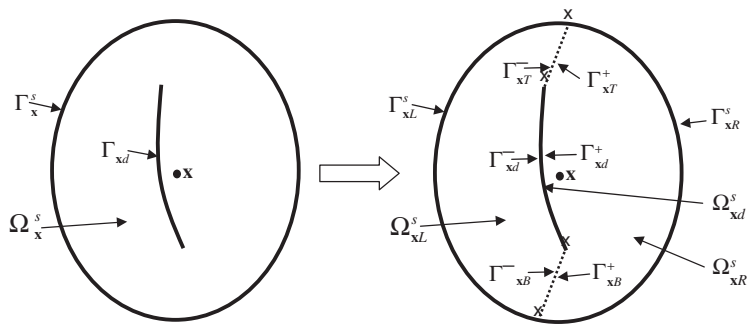


Fig. 2. Division of a local smoothing domain with a discontinuous line into three subdomains.

Since function w_l is continuous along $\Gamma_{\mathbf{x}T}$ and $\Gamma_{\mathbf{x}B}$, the summation of these two line-integration on $\Gamma_{\mathbf{x}Left}$ and $\Gamma_{\mathbf{x}Right}$ in Eq. (18) leads to the vanish of the line-integrations on $\Gamma_{\mathbf{x}T}$ and $\Gamma_{\mathbf{x}B}$, and we shall have

$$I_1 = \int_{\Gamma^s_{\mathbf{x}L}\cup\Gamma^-_{\mathbf{x}d}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\,\mathrm{d}\Gamma + \int_{\Gamma^s_{\mathbf{x}R}\cup\Gamma^+_{\mathbf{x}d}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma$$
$$= \int_{\Gamma^s_{\mathbf{x}}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma + \int_{\Gamma^-_{\mathbf{x}d}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma + \int_{\Gamma^+_{\mathbf{x}d}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma. \tag{19}$$

Integration I_2 in Eq. (18) becomes (in *Lebesgue* sense):

$$I_2 = \int_{\Omega^s_{\mathbf{x}L}} w_l(\xi)\frac{\partial \widehat{W}(\mathbf{x}-\xi)}{\partial x_i}\mathrm{d}\xi + \int_{\Omega^s_{\mathbf{x}R}} w_l(\xi)\frac{\partial \widehat{W}(\mathbf{x}-\xi)}{\partial x_i}\mathrm{d}\xi = \int_{\Omega^s_{\mathbf{x}}} w_l(\xi)\frac{\partial \widehat{W}(\mathbf{x}-\xi)}{\partial x_i}\mathrm{d}\xi. \tag{20}$$

In Eq. (18), however, I_3 needs to be approximated, because the derivative of w_l is not defined in $\Omega^s_{\mathbf{x}d}$. We then simply assume

$$I_3 = \int_{\Omega^s_{\mathbf{x}d}} \frac{\partial w_l(\xi)}{\partial x_j}\widehat{W}(\mathbf{x}-\xi)d\xi$$
$$\approx -\int_{\Gamma^+_{\mathbf{x}d}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma - \int_{\Gamma^-_{\mathbf{x}d}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma, \tag{21}$$

where the minus signs come from the fact that $\Gamma^+_{\mathbf{x}d}$ for subdomain $\Omega^s_{\mathbf{x}d}$ is the reverse in direction with respect to that for subdomain $\Omega^s_{\mathbf{x}R}$, and the same is also true for $\Gamma^-_{\mathbf{x}d}$. Equation (21) may be viewed as *gap smoothing* to be analyzed further later. Finally, submitting Eqs. (19), (20) and (21) into Eq. (18), we immediately have Eq. (17).

2.4.4. *On physical meaning of the gap smoothing*

Let's now examine the physical meaning of the "gap smoothing" approximation given in Eq. (21). Because the smoothing function is continuous on $\Gamma_{\mathbf{x}d}$, Eq. (21) becomes

$$I_3 \approx -\int_{\Gamma^+_{\mathbf{x}d}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma - \int_{\Gamma^-_{\mathbf{x}d}} w_l(\xi) n_i \widehat{W}(\mathbf{x}-\xi)\mathrm{d}\Gamma$$

$$= \left(\int_{\Gamma^+_{\mathbf{x}d}} w_l(\xi) n^-_{i\mathbf{x}d}\mathrm{d}\Gamma - \int_{\Gamma^-_{\mathbf{x}d}} w_l(\xi) n^-_{i\mathbf{x}d}\mathrm{d}\Gamma\right) \widehat{W}(\mathbf{x}-\xi_{\mathbf{x}d}). \tag{22}$$

For one-dimensional case along coordinate x as shown in Fig. 3, we shall have $n^-_{i\mathbf{x}d} = 1$. Further if $\widehat{W}(\mathbf{x}-\xi_{\mathbf{x}d}) = \frac{1}{l^s_{\mathbf{x}}}$ (see later for more precise definition for 2D cases) where $l^s_{\mathbf{x}}$ is the length of the smoothing domain $\Omega^s_{\mathbf{x}}$, Eq. (22) is simplified to

$$I_3 \approx \frac{w_l(\xi^+_{\mathbf{x}d}) - w_l(\xi^-_{\mathbf{x}d})}{l^s_{\mathbf{x}}}, \tag{23}$$

which means that we are approximating the "gradient" of the discontinuity of the function over the smoothed domain by simply dividing the discontinuous "gap" with the length of the smoothing domain. In other words, the gap over an infinitely thin discontinuous line is been "smeared" over the finite dimension of the smoothing domain. This analysis reveals intuitively that although we allow discontinuity in functions, the gap cannot be arbitrary. It cannot be finite and should approach zero when the dimension of the smoothing domains approaching zero. Since the smoothing domains are somehow tied with the filed nodes and hence with the cell dimension h, the dimension of the smoothing domain relates also to h. Such a "tie" cannot be too "loose", and thus we require the "minimum number of linearly independent smoothing domains" for a given set of cells/nodes as discussed in detail in [Liu (2008a, 2008b)]. Therefore, with such a tie, we now only require the gap approaching zero when h approaching zero. This can be always ensured when the nodal shape functions are properly created using the point interpolation method (PIM) or RPIM methods [Liu (2009)]. In summary, as long as a set of linearly independent nodal shape functions are created properly using the PIM for a given set of cells/nodes, the compatibility of these nodal shape functions is no longer a concern in our W^2 formulation based on the G space theory as demonstrated in [Liu (2008b)]. The stability and convergence are always ensured, even with the presence of the discontinuity in the assumed displacement functions.

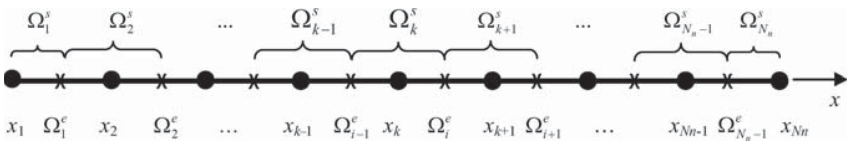


Fig. 3. Division of 1D domain with N_n nodes and $N_n - 1$ cells into a set of N_n local smoothing domains.

2.4.5. *Heaviside smoothing function*

For simplicity, we use in this paper the following special smoothing function that is a local constant (Heaviside type):

$$\widehat{W}(\mathbf{x}-\xi)=\bar{W}(\mathbf{x}-\xi)=\begin{cases}1/A_{\mathbf{x}}^{s} & \xi\in\boxed{\Omega}_{\mathbf{x}}^{s}\\ 0 & \xi\notin\boxed{\Omega}_{\mathbf{x}}^{s}\end{cases}, \tag{24}$$

where $A_{\mathbf{x}}^{s}=\int_{\Omega_{\mathbf{x}}^{s}}d\Omega$ is the area of smoothing domain the point at $\mathbf{x}$. It is clear that $\widehat{W}(\mathbf{x}-\xi)$ given above satisfies the conditions of unity, positivity and decay. In this work we use *stationary* smoothing domains that are fixed for a point of interest. We do not allow the smoothing domains to overlap: $\boxed{\Omega}=\bigcup_{k=1}^{N_s}\boxed{\Omega}_{k}^{s}$ where Ω_k^s is a smoothing domain bounded by Γ_k^s or point at $\mathbf{x}_k$, and N_s is the number of total smoothing domains as shown in Fig. 4. In this case, we have:

$$\overline{\frac{\partial w_l}{\partial x_i}}(\mathbf{x})=\overline{\frac{\partial w_l}{\partial x_i}}(\mathbf{x}_k)=\begin{cases}\dfrac{1}{A_s}\displaystyle\int_{\Omega_k^s}\frac{\partial w_l}{\partial x_i}d\Omega & \\ \quad=\underbrace{\dfrac{1}{A_k^s}\displaystyle\int_{\Gamma_k^s}w_l(\mathfrak{S})n_i\mathrm{d}\mathfrak{S}}_{\text{constant in }\Omega_k^s}, & w_l(\mathbf{x})\text{ is continous in }\Omega_k^s\\ \underbrace{\dfrac{1}{A_k^s}\displaystyle\int_{\Gamma_k^s}w_l(\mathfrak{S})n_i\mathrm{d}\mathfrak{S}}_{\text{constant in }\Omega_k^s}, & w_l(\mathbf{x})\text{ is discontinous in }\Omega_k^s\end{cases} \tag{25}$$

$$l=1,2;\quad i=1,2;\quad \forall\mathbf{x}\in\Omega_k^s$$

In carrying out the line integrations, we simply use the standard Gauss integration that are widely used in standard FEM [Liu and Quek (2003)].

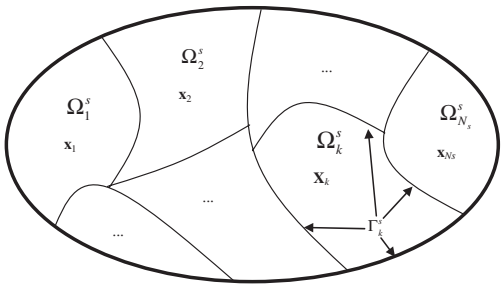


Fig. 4. Division of problem domain Ω into non-overlapping stationary smoothing domains Ω_i^s for $\mathbf{x}_i$ bounded by Γ_k^s. The smoothing domain is also used as the basis for integration.

2.4.6. *On physical meaning of the gradient approximation*

Let's now examine the physical meaning of Eq. (25). For easy comprehension, we consider again 1D case with coordinate x as shown in Fig. 3. In such cases, $n_i = 1$, and Eq. (25) can be rewritten as

$$\overline{\frac{\partial w_l}{\partial x_i}}(\mathbf{x}_k) = \frac{1}{A_k^s}\int_{\Gamma_k^s} w_l(\mathbb{S})n_i \mathrm{d}\mathbb{S} = \frac{w_l(x_{k+\frac{1}{2}}) - w_l(x_{k-\frac{1}{2}})}{l_k^s}, \tag{26}$$

which is the "averaged" gradient of the function evaluated using the function values at two ends (boundary) of the smoothing domain divided by the length (dimension) of the smoothing domain.

Equation (26) can also be viewed as the well-known central difference scheme performed over the smoothing domain. Therefore, the gradient smoothing domain technique can be used to create finite difference formulation as long as proper different types of smoothing domains can be used in a proper manner [Liu and Xu (2008)]. The so-called gradient smoothing method (GSM) is one typical such a method developed for efficient analysis of fluid dynamic problems using general unstructured triangular grids [Liu and Xu (2008)]. The GSM is found particularly efficient using unstructured cells and hence works very well even for adaptive analysis of CFD problems [Xu, Liu and Tani (2009)].

2.4.7. *On the generalized gradient approximation*

We now ready to discuss about the physical meaning and numerical implications of the generalized gradient smoothing defined in the second equation of Eq. (25). It is seen that the approximation consists two items: gradient approximation for the continuous portions of the function in the manner discussed in Sec. 2.4.6 and the gradient approximation for the discontinuity of the function via gap smoothing. All these approximations provide the first order approximation for the function and are capable of producing the gradient of the function (continuous or discontinuous). Therefore, as long as a set of linearly independent and linearly complete nodal shape functions can be used to create functions in a proper G space for a discrete model, and a proper norms can be defined for W^2 formulation [Liu (2008b)] or a proper use of smoothing domains for strong formulation [Liu and Xu (2008)]. To ensure the stability, the model can always pass the standard patch tests [Liu (2008b)]. This analysis shows also that even higher order interpolations can be used, the accuracy of the solution may not be necessarily increased accordingly. This is also observed in [Liu (2008b)]. To have a higher order in solution accuracy, higher order of smoothing functions and alternative treatments on the discontinuous gap may be needed. In summary, the W^2 formulation works for all order of piecewisely continuous linearly independent nodal shape functions, and there is no worries about continuity effects on the stability and convergence of the solution of the discretized model, but it works most efficiently for lower (linear or bilinear) interpolation models, and in particular models of linear triangular cells/elements.

3. G Spaces

3.1. *General notation of derivatives*

For the convenience of space definitions, we first define the notation of differentiations:

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \tag{27}$$

where d is the dimension of the problem domain, α is n-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_d)$, and $|\alpha| = \sum_{i=1}^{d} \alpha_i$.

3.2. *Briefing on H spaces*

Consider a domain Ω discretized by for example triangulation with N_e non-overlapping subdomains $\boxed{\Omega} = \bigcup_{i=1}^{N_e} \boxed{\Omega}_i^e$ with a set of N_n nodes, and N_c^{Γ} line segments Γ_i^c $(i = 1, \ldots, N_c^{\Gamma})$ related to the cell (meshfree settings) or element (FEM settings) division. In the FEM based on weak forms for stable and convergent solutions, we had to be very "choosy": the assumed displacement functions have to be in a proper Hilbert space that can be defined in general as: for a non-negative integer m,

$$\mathbb{H}^m(\Omega) = \{v | D^{\alpha} v \in \mathbb{L}^2(\Omega), \forall |\alpha| \leq m\}, \tag{28}$$

where $\mathbb{L}^2(\Omega)$ denotes the Lebesgue space. The $\mathbb{H}^m(\Omega)$ hosts all functions whose derivatives up to mth order are all square integrable. The associated inner product is given by

$$(w, v) = \sum_{|\alpha| \leq m} \int_{\Omega} (D^{\alpha} w) \cdot (D^{\alpha} v) \mathrm{d}\Omega, \tag{29}$$

and induced (full) norm

$$\|v\|_{\mathbb{H}^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_{\Omega} |D^{\alpha} v|^2 \mathrm{d}\Omega \right)^{1/2}, \tag{30}$$

as well as the semi-norm (that includes only the mth derivative):

$$|v|_{\mathbb{H}^m(\Omega)} = \left(\int_{\Omega} |D^{\alpha} v|^2 \, \mathrm{d}\Omega \right)^{1/2}. \tag{31}$$

Essentially, the $\mathbb{H}^m$ full norm measures the first m (including zero) derivatives of v in the $\mathbb{L}^2$ norm. For problems of 2D solid mechanics, the H^1 for space should be used [Hughes (1987)]:

$$\mathbb{H}^1(\Omega) = \{v | v \in \mathbb{L}^2(\Omega), \partial v / \partial x_i \in \mathbb{L}^2(\Omega), i = 1, 2\}, \tag{32}$$

and in particular $\mathbb{H}^1_0(\Omega) = \{v \in \mathbb{H}^1(\Omega) | v_i = 0 \text{ on } \Gamma_D\}$. The H^1 full norm is defined as

$$\|w\|^2_{\mathbb{H}^1(\Omega)} = \underbrace{\int_\Omega w^2 \mathrm{d}\Omega}_{\|w\|^2_{\mathbb{L}^2(\Omega)}} + \underbrace{\int_\Omega (\nabla w) \cdot (\nabla w) \mathrm{d}\Omega}_{|w|^2_{\mathbb{H}^1(\Omega)} = \|\nabla w\|^2_{\mathbb{L}^2(\Omega)}}, \tag{33}$$

where

$$\nabla w = \begin{pmatrix} \dfrac{\partial w}{\partial x_1} & \dfrac{\partial w}{\partial x_2} \end{pmatrix}. \tag{34}$$

The $\mathrm{H}^1(\Omega)$ semi-norm becomes

$$|w|^2_{\mathbb{H}^1(\Omega)} = \underbrace{\int_\Omega (\nabla w) \cdot (\nabla w) \mathrm{d}\Omega}_{\|\nabla w\|^2_{\mathbb{L}^2(\Omega)}}. \tag{35}$$

An a subspace in $\mathbb{H}^1$ space created using interpolation techniques that ensures compatibility can then be defined as

$$\mathbb{H}^1_h(\Omega) = \{v \in \mathbb{H}^1(\Omega) | v(\mathbf{x}) = \boldsymbol{\varphi}^H(\mathbf{x})\mathbf{d}, \mathbf{d} \in \mathbb{R}^{N_n}\}, \tag{36}$$

where $\boldsymbol{\varphi}^H(\mathbf{x})$ is the matrix of all the (compatible) nodal shape functions constructed using an FEM model, and can be written as

$$\boldsymbol{\varphi}^H(\mathbf{x}) = [\phi^H_1(\mathbf{x}) \quad \phi^H_2(\mathbf{x}) \quad \cdots \quad \phi^H_{N_n}(\mathbf{x})]. \tag{37}$$

Because $\mathbb{H}^1_h(\Omega)$ is a linear space, each of the nodal shape functions $\phi^H_i(\mathbf{x})$ must also be in $\mathbb{H}^1_h(\Omega)$. The linearly independence of shape functions $\phi^H_i(\mathbf{x}), (i = 1, 2, \ldots, N_n)$ are ensured by a standard FEM procedure (element-based and proper mapping). In Equation (36) $\mathbf{d}$ is the vector of all the nodal functions values given in the form of

$$\mathbf{d} = \{v_1 \quad v_2 \quad \cdots \quad v_{N_n}\}^{\mathrm{T}}. \tag{38}$$

Since the values at each node can change independently, we have $\mathbf{d} \in \mathbb{R}^{N_n}$ where $\mathbb{R}^{N_n}$ stands for a real field of N_n dimensions.

Because $\mathbb{H}^1_h$ is constructed in a discrete form with finite dimensions, it is marked with a subscript "h". Functions in $\mathbb{H}^1_h$ that satisfy the essential (displacement) boundary conditions form a space:

$$\mathbb{H}^1_{h,0}(\Omega) = \{v \in \mathbb{H}^1_h(\Omega) | v = 0 \text{ on } \Gamma_u\}. \tag{39}$$

An $\mathbb{H}^1_h$ space is indeed very exclusive, and the methods that can be used to create functions in an $\mathbb{H}^1_h$ space are very much limited: FEM technique and the MLS approximation.

3.3. *Definitions for G spaces*

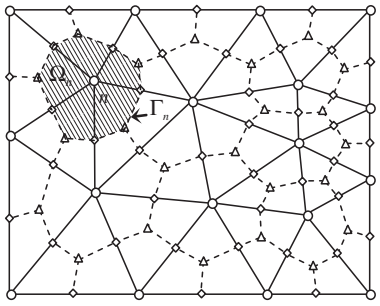
3.3.1. *Smoothing domain creation*

We next divide, in a basically independent way, the domain Ω into N_s non-overlapping and no-gap subdomains called smoothing domains: $\boxed{\Omega} = \bigcup_{k=1}^{N_s} \boxed{\Omega}_k^s$ with N_s^Γ line interfaces Γ_i^s $(i = 1, \ldots, N_s^\Gamma)$ between the smoothing domains. The division of Ω into $\boxed{\Omega}_k^s$ is performed in such a way that the interfaces Γ_i^s do not share any *finite* portion of the interfaces Γ_i^c on which the function is not square integrable. The interfaces of $\boxed{\Omega}_k^s$ can go across Γ_i^c. Only when the function is continuous on Γ_i^c, the sharing of the interfaces of $\boxed{\Omega}_k^s$ and $\boxed{\Omega}_l^e$ may be permitted. A typical division of domain is given in Fig. 5 for 2D domains, where node-based smoothing domains are created upon a set of triangular background cells. Figure 3 shows an example of such a division for a 1D domain.

3.3.2. *General definition of G space*

In this paper a general expression of G spaces is given as follows. For a non-negative m:

$$\mathbb{G}_h^m(\Omega) = \left\{ \begin{array}{l} v \Big| \; v(\mathbf{x}) = \sum_{n=1}^{N_n} \phi_n(\mathbf{x}) d_n = \boldsymbol{\varphi}(\mathbf{x})\mathbf{d}, \mathbf{d} \in \mathbb{R}^{N_n} \\ \quad D^\alpha v \in \mathbb{L}^2(\Omega), \forall \alpha : |\alpha| \le (m-1), \\ \quad \sum_{k=1}^{N_s} \left(\int_{\Gamma_k^s} (D^\alpha v) n_i \mathrm{d}s \right)^2 > 0 \Leftrightarrow v \ne c \in \mathbb{R}; i = 1, \cdots, d, \forall \alpha : |\alpha| \le (m-1) \end{array} \right. , \tag{40}$$



○ Field nodes △ Centroid of triangle ◇ Mid-edge-point

Fig. 5. A node-based smoothing domains for G space functions. Triangular cells/elements (bounded by solid lines) are created using the filed nodes; and the smoothing domain (bounded by dashed lines) for a node is created by connecting the centroids with the mid-edge-points of the surrounding triangles of a node. Γ_n does not share any finite portion of any internal edges of the triangular cells/elements.

where $\mathbf{d} = \{d_1 \quad d_2 \quad \cdots \quad d_{N_n}\}^T$ is the vector of nodal function values, and $\boldsymbol{\varphi}(\mathbf{x})$ is the matrix of nodal shape functions constructed using a general point interpolation method [Liu (2009)], and can be written as

$$\boldsymbol{\varphi}(\mathbf{x}) = [\phi_1(\mathbf{x}) \quad \phi_2(\mathbf{x}) \quad \cdots \quad \phi_{N_n}(\mathbf{x})]. \tag{41}$$

It is observed that the G space is a set of linear independent shape functions. The derivatives of the functions up to the $(m-1)$th orders are square integrable in Ω. Because of the discretized nature of a G space, it is marked with a subscript "h". The major difference between a G_h space and the corresponding *discrete* Hilbert space or H_h space is that the H space requires $D^\alpha v \in \mathbf{L}^2(\Omega)$ for $|\alpha| = m$, but in the G_h space we require only $D^\alpha v \in \mathbf{L}^2(\Omega)$ for $|\alpha| = (m-1)$. Therefore, the requirement on function is now further weakened upon the already weakened requirement for functions in an H space, and hence a G space can be viewed as a space of a set of functions with weakened weak (W^2) requirements. It is therefore clear now that a $\mathbb{H}_h^m(\Omega)$ is also a $\mathbb{G}_h^m(\Omega)$ space: any function in $\mathbb{H}_h^m(\Omega)$ is surely qualified as a member in $\mathbb{G}_h^m(\Omega)$.

The inner product associated with a $\mathbb{G}_h^m(\Omega)$ space is then defined as

$$(w, v)_{\mathbb{G}^m(\Omega)} = \sum_{|\alpha| \le m-1} \left(\int_\Omega D^\alpha w \cdot D^\alpha v \mathrm{d}\Omega \right) + \sum_{|\alpha|=m} \sum_{k=1}^{N_s} A_k^s \overline{D^\alpha w} \cdot \overline{D^\alpha v}. \tag{42}$$

The inner product induced (full) norm is next defined as

$$\|w\|_{\mathbb{G}^m(\Omega)} = \left[\sum_{|\alpha| \le m-1} \left(\int_\Omega |D^\alpha w|^2 \mathrm{d}\Omega \right) + \sum_{|\alpha|=m} \left(\sum_{k=1}^{N_s} A_k^s \left| \overline{D^\alpha w} \right|^2 \right) \right]^{1/2}. \tag{43}$$

The $\mathbb{G}_h^m(\Omega)$ semi-norm is finally defined using only the smoothed αth derivatives:

$$|w|_{\mathbb{G}^m(\Omega)} = \left[\sum_{|\alpha|=m} \left(\sum_{k=1}^{N_s} A_k^s \left| \overline{D^\alpha w} \right|^2 \right) \right]^{1/2}. \tag{44}$$

In this work we use only the $\mathbb{G}_h^1$ spaces, and therefore more details are given in the following sections.

3.3.3. *G^1 space and norms*

The $\mathbb{G}_h^1$ that is relevant to this work can be then defined as follows.

$$\mathbb{G}_h^1(\Omega) = \left\{ \begin{array}{l} v \Big| \; v(\mathbf{x}) = \displaystyle\sum_{n=1}^{N_n} \phi_n(\mathbf{x}) d_n = \boldsymbol{\varphi}(\mathbf{x})\mathbf{d}, \; \mathbf{d} \in \mathbb{R}^{N_n} \\ \quad v \in \mathbb{L}^2(\Omega), \\ \quad \displaystyle\sum_{k=1}^{N_s} \left(\int_{\Gamma_k^s} v(s) n_i \mathrm{d}s \right)^2 > 0 \Leftrightarrow \; v \neq c \in \mathbb{R}; i = 1, \ldots, d \end{array} \right\} \tag{45}$$

In creating functions in $\mathbb{G}_h^1$ spaces, we do not restrict on how these shape functions are created, as long as they satisfy the following conditions:

(1) *Linear independency condition*: all these nodal shape functions are linearly independent over Ω and hence are capable to form a basis.
(2) *Bound condition*: all the functions constructed using these shape function must be square integrable over the problems domain. This is to ensure the convergence of a numerical model to be created.
(3) *Positivity conditions*: there exist a division of Ω_k^s such that $\sum_{k=1}^{N_s} (\int_{\Gamma_k^s} v(\mathcal{S}) n_i \mathrm{d}\mathcal{S})^2 > 0, \Leftrightarrow v \neq c \in \mathbb{R}$, $\forall \mathbf{d} \in \mathbb{R}^{N_n}$ and $i = 1, \ldots, d$. This (together with the linearly independent condition) is to ensure the stability of a numerical model to be created.

When PIM or RPIM shape functions [Liu and Gu (2005); Liu (2009)] are used, the functions constructed will in general not be continuous over the entire problem domain and hence are not compatible. Such an interpolant is not in an $\mathbb{H}_h^1$ space, but in a $\mathbb{G}_h^1$ space, because all these above-listed three conditions can be satisfied. Note that since $\phi_i(\mathbf{x})$, $(i = 1, 2, \ldots, N_n)$ are constructed using nodes selected using a cell-based T-Scheme [Liu (2009)] or edge-based T-Scheme [Liu and Zhang (2009)], and at least the three nodes of any home cell are always used. Hence, all PIM shape functions and RPIM shape functions with linear polynomial basis constructed using a T-scheme, which ensures the model passing the standard patch tests.

The major difference between a $\mathbb{G}_h^1$ space and $\mathbb{H}_h^1$ space is that the $\mathbb{H}_h^1$ space requires the first gradient of the function square integrable, but in the $\mathbb{G}_h^1$ space we require only the function itself square integrable. Therefore, the requirement on function is now further weakened upon the already weakened requirement for functions in an $\mathbb{H}_h^1$ space, and hence a $\mathbb{G}_h^1$ space can be viewed as space of a set of functions with weakened weak (W^2) requirements on continuity. In an $\mathbb{H}_h^1$ space, the bound condition is achieved by the imposing the smoothness upon the first derivatives of the function to be square integrable, while in the $\mathbb{G}_h^1$ space, it is controlled by imposing the smoothness only on the function to be square integrable (with a proper construction of smoothing domains). The stability is automatically ensured for functions in an $\mathbb{H}_h^1$ space as long as the smoothness is satisfied, due to the Poincare-Friedrichs inequality. The stability in the $\mathbb{G}_h^1$ space, however, is ensured by imposing the positivity condition. Because a member in a $\mathbb{G}_h^1$ space is also a member of the $\mathbb{L}^2$ space, therefore a $\mathbb{G}_h^1$ space is a subspace of $\mathbb{L}^2$ space: $\mathbb{G}_h^1(\Omega) \subset \mathbb{L}^2(\Omega)$. A $\mathbb{G}_h^1$ space is indeed very accommodating and inclusive, and hence shall have much wider applications in the formulation of various numerical methods.

3.3.4. *Normed or un-normed G spaces*

G space can either be normed or un-normed. Un-normed G spaces are used for the strain-constructed Galerkin (or SC-Galerkin) models [Liu (2009)], where admissible conditions for the constructed strains are defined properly in a separated manner.

The normed G space is used to formulate the generalized smoothed Galerkin (or GS-Galerkin) models that are special cases of W^2 formulations. Normed G spaces require a proper construction of smoothing domains following the rules detailed in Sec. 3.3.1.

Note that un-normed G spaces can be used for establishing strong form meshfree methods by, for example, simple collocation. In such cases, the stability is left "uncontrolled" when the assumed functions are used to create a discrete model, and we need additional procedure such as the regularization techniques [Liu and Kee (2006); Kee, Liu and Lu (2007)] or proper used of different types of smoothing domains [Liu and Xu (2008); Xu, Liu and Tani (2009)] to restore the stability.

3.3.5. *G^1 norms for 1D scalar fields*

For normed $\mathbb{G}^1_h$ spaces, the norms are *induced* from the inner products defined as follows for various cases. The associated inner product is given by:

$$(w,v)_{\mathbb{G}^1(\Omega)} = \int_\Omega wv\mathrm{d}\Omega + \sum_{k=1}^{N_s} A_k^s \overline{w'}\cdot\overline{v'} = \underbrace{\int_\Omega wv\mathrm{d}\Omega}_{(w,v)_{\mathbb{L}^2(\Omega)}} + \underbrace{\sum_{k=1}^{N_s} A_k^s \bar{g}(w)\bar{g}(v)}_{(\overline{w'},\overline{v'})_{\mathbb{L}^2(\Omega)}}. \tag{46}$$

Note the summation is possible because the division of Ω into Ω_i^s is performed in such a way that the interfaces Γ_i^s of Ω_i^s do not share any *finite* portion of the interfaces Γ_i^c on which the function is not square integrable: no energy loss in the interface of the smoothing domains. In Eq. (46) the (approximated) smoothed gradient is denoted as

$$\overline{w'} = \overline{\frac{\partial w}{\partial x}} = \frac{1}{A_k^s}\int_{\Gamma_k^s} w(\mathcal{S})n_x\mathrm{d}\mathcal{S} = \underbrace{\frac{1}{l_k^s}(w_{k+\frac{1}{2}} - w_{k-\frac{1}{2}})}_{=\bar{g},\ \text{constant in } \Omega_k^s} = \bar{g}(w), \tag{47}$$

where $\bar{g}(w)$ denotes the smoothed derivatives of w with respect to x, and the smoothing domains Ω_k^s is "centered" at $x_s(=x_k$, in this 1D case) is bounded by $x_{k-\frac{1}{2}}$ and $x_{k+\frac{1}{2}}$, as shown in Fig. 3. The G^1 semi-norm is next defined as

$$|w|^2_{\mathbb{G}^1(\Omega)} = \sum_{k=1}^{N_s} A_k^s|\overline{w'}|^2 = \underbrace{\sum_{k=1}^{N_s} l_k^s \bar{g}^2(w)}_{(\overline{w'},\overline{w'})}, \tag{48}$$

and the G^1 full norm becomes

$$\|w\|^2_{\mathbb{G}^1(\Omega)} = \underbrace{\int_\Omega w^2\mathrm{d}\Omega}_{(w,w)=\|w\|^2_{L^2}} + \underbrace{|w|^2_{\mathbb{G}^1(\Omega)}}_{(\overline{w'},\overline{w'})=|w|^2_{\mathbb{G}^1(\Omega)}} = \|w\|^2_{L^2} + |w|^2_{\mathbb{G}^1(\Omega)}, \tag{49}$$

which is induced from the inner product defined in Eq. (46).

3.3.6. G^1 *norms for 2D scalar fields*

For 2D problems, division of problem domain into smoothing domains is in general shown in Fig. 4 with at least the minimum number of linearly independent smoothing domains [Liu (2008a, 2008b)]. The proven models are the node-based as shown

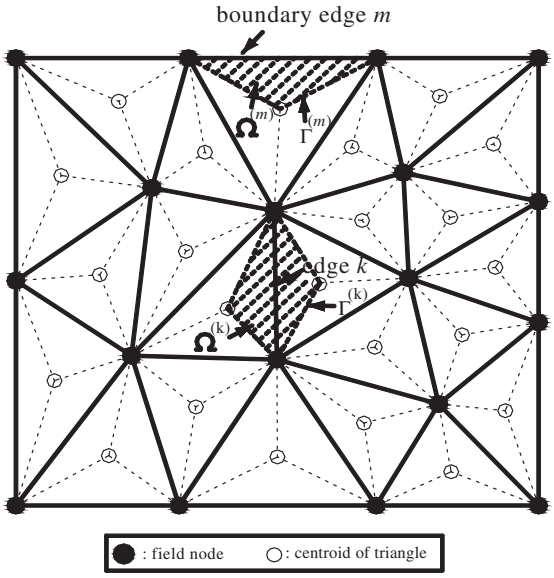


Fig. 6. A edge-based smoothing domains for G space functions. Triangular cells/elements (bounded by solid lines) are created using the filed nodes; and the smoothing domain (bounded by dashed lines) for a node is created by connecting the centroids with the mid-edge-points of the surrounding triangles of a node. Γ_k^s does not share any finite portion of any internal edges of the triangular cells/elements.

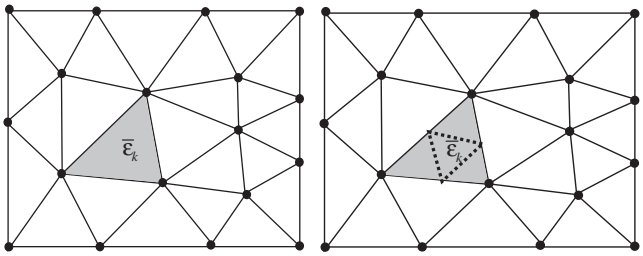


Fig. 7. Cell-based smoothing domains for $\mathbb{G}_h^1$ space functions. Triangular cells/elements (bounded by solid lines) are created using the filed nodes; and the smoothing domains are either the triangular cell or a sub-triangular-cell obtained by further dividing the triangular cell. The PIM or RPIM shape functions are formed associated with the boundaries of the smoothing cells [Liu and Zhang (2009)].

in Fig. 5, edge-based shown in Fig. 6 and cell-based (Fig. 7). Using such a set of smoothing domains, the associated inner product is given by:

$$\begin{aligned}(w,v)_{\mathbb{G}^1(\Omega)} &= \int_\Omega wv\mathrm{d}\Omega + \sum_{k=1}^{N_s} A_k^s \overline{\nabla w}\cdot\overline{\nabla v} \\ &= \underbrace{\int_\Omega wv\mathrm{d}\Omega}_{(w,v)_{\mathbb{L}^2(\Omega)}} + \underbrace{\sum_{k=1}^{N_s} A_k^s(\bar{g}_1(w)\bar{g}_1(v)+\bar{g}_2(w)\bar{g}_2(v))}_{(\overline{\nabla w},\overline{\nabla v})_{\mathbb{L}^2(\Omega)}},\end{aligned} \tag{50}$$

where the (approximated) smoothed gradient is denoted as

$$\begin{aligned}\overline{\nabla w} &= \begin{pmatrix}\overline{\dfrac{\partial w}{\partial x_1}} & \overline{\dfrac{\partial w}{\partial x_2}}\end{pmatrix} \\ &= \begin{pmatrix}\underbrace{\dfrac{1}{A_k^s}\displaystyle\int_{\Gamma_k^s} w(\mathfrak{S})n_1\mathrm{d}\mathfrak{S}}_{=\bar{g}_1,\ \text{constant in } \Omega_k^s} & \underbrace{\dfrac{1}{A_k^s}\displaystyle\int_{\Gamma_k^s} w(\mathfrak{S})n_2\mathrm{d}\mathfrak{S}}_{=\bar{g}_2,\ \text{constant in } \Omega_k^s}\end{pmatrix} \\ &= (\bar{g}_1(w) \quad \bar{g}_2(w)),\end{aligned} \tag{51}$$

where $\bar{g}_i(w)$ denotes the smoothed derivatives of w with respect to x_i.

The G^1 (Ω) semi-norm is next defined as

$$|w|^2_{\mathbb{G}^1(\Omega)} = \sum_{n=1}^{N_s} A_n^s|\overline{\nabla w}|^2 = \underbrace{\sum_{n=1}^{N_s} A_n^s(\bar{g}_1^2(w)+\bar{g}_2^2(w))}_{(\overline{\nabla w},\overline{\nabla w})}, \tag{52}$$

and the G^1 full norm becomes

$$\|w\|^2_{\mathbb{G}^1(\Omega)} = \underbrace{\int_\Omega w^2\mathrm{d}\Omega}_{(w,w)=\|w\|^2_{L^2}} + \underbrace{|w|^2_{\mathbb{G}^1(\Omega)}}_{(\overline{\nabla w},\overline{\nabla w})} = \|w\|^2_{L^2} + |w|^2_{\mathbb{G}^1(\Omega)}, \tag{53}$$

which is induced from the inner product Eq. (50). The definitions for 3D scalar fields are natural extension and hence are omitted here.

3.3.7. *G^1 norms for 2D vector fields*

For vector fields, we need to use vectors of functions. The division of problem domains into smoothing domains is similar as that for the 2D scalar field discussed in the previous subsection. When the function has two components, we should have $w = (w_1 \quad w_2)$ where $w_1, w_2 \in \mathbb{G}_h^1$ are the two component functions. In this case,

we have the smoothed gradient in the following form.

$$\overline{\nabla \mathbf{w}} = \begin{pmatrix} \overline{\dfrac{\partial w_1}{\partial x_1}} & \overline{\dfrac{\partial w_1}{\partial x_2}} \\ \overline{\dfrac{\partial w_2}{\partial x_1}} & \overline{\dfrac{\partial w_2}{\partial x_2}} \end{pmatrix} = \begin{pmatrix} \underbrace{\dfrac{1}{A_k^s}\int_{\Gamma_k^s} w_1(\mathfrak{S}) n_1 \mathrm{d}\mathfrak{S}}_{=\bar{g}_{11},\ \text{constant in } \Omega_k^s} & \underbrace{\dfrac{1}{A_k^s}\int_{\Gamma_k^s} w_1(\mathfrak{S}) n_2 \mathrm{d}\mathfrak{S}}_{=\bar{g}_{12},\ \text{constant in } \Omega_k^s} \\ \underbrace{\dfrac{1}{A_k^s}\int_{\Gamma_k^s} w_2(\mathfrak{S}) n_1 \mathrm{d}\mathfrak{S}}_{=\bar{g}_{21},\ \text{constant in } \Omega_k^s} & \underbrace{\dfrac{1}{A_k^s}\int_{\Gamma_k^s} w_2(\mathfrak{S}) n_2 \mathrm{d}\mathfrak{S}}_{=\bar{g}_{22},\ \text{constant in } \Omega_k^s} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{g}_{11}(w_1) & \bar{g}_{12}(w_1) \\ \bar{g}_{21}(w_2) & \bar{g}_{22}(w_2) \end{pmatrix}, \tag{54}$$

where $\bar{g}_{ij}(w)$ denotes the smoothed derivatives of w_i with respect to x_j. We notice here that the (smoothed) gradient is now a matrix, and hence there can be many *equivalent* ways to define the associated inner product. In this work, we decide to have the definition associated with the type of physical problems to be studied for convenience of proving necessary theories for that type of the problems. Considering 2D solid mechanics problems, we define the associated inner product in the form of

$$(\mathbf{w}, \mathbf{v})_{\mathbb{G}^1(\Omega)} = \underbrace{\int_\Omega (w_1 v_1 + w_2 v_2) \mathrm{d}\Omega}_{(w,v)} + \underbrace{\sum_{k=1}^{N_s} A_k^s \begin{bmatrix} \bar{g}_{11}(w_1)\bar{g}_{11}(v_1) + \bar{g}_{22}(w_2)\bar{g}_{22}(v_2) \\ + (\bar{g}_{12}(w_1) + \bar{g}_{21}(w_2))(\bar{g}_{12}(v_1) + \bar{g}_{21}(v_2)) \end{bmatrix}}_{(\overline{\nabla \mathbf{w}}, \overline{\nabla \mathbf{v}})}. \tag{55}$$

The induced $\mathrm{G}^1(\Omega)$ semi-norm is first defined as

$$|\mathbf{w}|^2_{\mathbb{G}^1(\Omega)} = \underbrace{\sum_{k=1}^{N_s} A_k^s (\bar{g}_{11}^2(w_1) + \bar{g}_{22}^2(w_2) + (\bar{g}_{12}(w_1) + \bar{g}_{21}(w_2))^2)}_{(\overline{\nabla \mathbf{w}}, \overline{\nabla \mathbf{w}})}. \tag{56}$$

It is clear that in our definition of the inner product and hence the induced the semi-norm we have intentionally related to the strain components, and hence the L^2 norm of the vector of strains.

The associated G^1 full norm can now be defined as

$$\|\mathbf{w}\|^2_{\mathbb{G}^1(\Omega)} = \underbrace{\int_\Omega (w_1^2 + w_2^2)\, \mathrm{d}\Omega}_{(\mathbf{w},\mathbf{w}) = \|\mathbf{w}\|^2_{L^2}} + \underbrace{|\mathbf{w}|^2_{\mathbb{G}^1(\Omega)}}_{(\overline{\nabla \mathbf{w}}, \overline{\nabla \mathbf{w}})} = \|\mathbf{w}\|^2_{L^2} + |\mathbf{w}|^2_{\mathbb{G}^1(\Omega)}. \tag{57}$$

3.3.8. *G^1 norms for 3D vector fields*

For three-dimensions (3D) problems, the division of problem domain into smoothing domains is similar to those in 2D but with one more dimension extension. Such an

extension makes the division much more difficult. The proven models are node-based and face-based [Liu, Nguyen-Thoi and Lam (2008); Zhang and Liu *et al.* (2007); Wu *et al.* (2008)]. For vector fields with three-component functions in 3D space, such as the 3D solid mechanics problems, we shall have $w = (w_1 \quad w_2 \quad w_3)$ where $w_1, w_2, w_3 \in \mathbb{G}_h^1$. In this case we define, naturally, the associated inner product as

$$(\mathbf{w}, \mathbf{v})_{\mathbb{G}^1(\Omega)} = \int_\Omega (w_1 v_1 + w_2 v_2 + w_3 v_3) \mathrm{d}\Omega + \sum_{k=1}^{N_s} A_k^s \begin{bmatrix} \bar{g}_{11}(w_1)\bar{g}_{11}(v_1) + \bar{g}_{22}(w_2)\bar{g}_{22}(v_2) + \bar{g}_{33}(w_3)\bar{g}_3(v_3) \\ + (\bar{g}_{12}(w_1) + \bar{g}_{21}(w_2))(\bar{g}_{12}(v_1) + \bar{g}_{21}(v_2)) \\ + (\bar{g}_{13}(w_1) + \bar{g}_{31}(w_3))(\bar{g}_{13}(v_1) + \bar{g}_{31}(v_3)) \\ + (\bar{g}_{23}(w_2) + \bar{g}_{32}(w_3))(\bar{g}_{23}(v_2) + \bar{g}_{32}(v_3)) \end{bmatrix}. \tag{58}$$

The associated $\mathrm{G}^1(\Omega)$ semi-norm can be defined as

$$|\mathbf{w}|^2_{\mathbb{G}^1(\Omega)} = \sum_{k=1}^{N_s} A_k^s \begin{pmatrix} \bar{g}_{11}^2(w_1) + \bar{g}_{22}^2(w_2) + \bar{g}_{33}^2(w_3) \\ + (\bar{g}_{12}(w_1) + \bar{g}_{21}(w_2))^2 \\ + (\bar{g}_{13}(w_1) + \bar{g}_{31}(w_3))^2 \\ + (\bar{g}_{23}(w_2) + \bar{g}_{32}(w_3))^2 \end{pmatrix}, \tag{59}$$

and the G^1 full norm is given by

$$\|\mathbf{w}\|^2_{\mathbb{G}^1(\Omega)} = \int_\Omega (w_1^2 + w_2^2 + w_3^2)\mathrm{d}\Omega + |\mathbf{w}|^2_{\mathbb{G}^1(\Omega)}. \tag{60}$$

We finally denote a space for functions that are fixed on the Dirichlet boundaries and hence the functions cannot "float".

$$\mathbb{G}^1_{h,0} = \{v \in \mathbb{G}^1_h(\Omega) | v_i = 0 \text{ on } \Gamma_u\}. \tag{61}$$

3.4. *Basic properties*

Because the normed $\mathbb{G}_h^1$ spaces are defined in the above-mentioned "unusual" manner, we have to show that they possess all the necessary basic properties. Here we discuss only normed $\mathbb{G}_h^1$ spaces.

3.4.1. *Linearity*

Remark 3.1. A $\mathbb{G}_h^1$ space is a *linear* space. To show this, consider any two functions $w, v \in \mathbb{G}_h^1$. We shall have

$$\begin{aligned} w(\mathbf{x}) &= \boldsymbol{\varphi}(\mathbf{x})\mathbf{d}_w, \mathbf{d}_w \in \mathbb{R}^{N_n} \\ v(\mathbf{x}) &= \boldsymbol{\varphi}(\mathbf{x})\mathbf{d}_v, \mathbf{d}_v \in \mathbb{R}^{N_n} \end{aligned}. \tag{62}$$

The addition w and v becomes

$$w(\mathbf{x}) + v(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x}) \underbrace{(\mathbf{d}_w + \mathbf{d}_v)}_{\in \mathbb{R}^{N_n}}, \tag{63}$$

which must also be in $\mathbb{G}_h^1$, because $\mathbb{R}^{N_n}$ is a linear space: $(\mathbf{d}_w + \mathbf{d}_v) \in \mathbb{R}^{N_n}$. Following exactly the same argument, we shall have that for $\forall w \in \mathbb{G}_h^1$ and $\forall \alpha \in \mathbb{R}$, $\alpha w \in \mathbb{G}_h^1$.

3.4.2. *Positivity*

Remark 3.2. A function in a $\mathbb{G}_h^1$ space is always non-negative in norm:

$$\|w\|_{\mathbb{G}^1} \geq 0, \quad \forall w \in \mathbb{G}_h^1, \tag{64}$$

and a nonzero function in a $\mathbb{G}_h^1$ space is always strictly positive in norm:

$$\|w\|_{\mathbb{G}^1} > 0, \quad \forall w \in \mathbb{G}_h^1, \quad w \neq 0. \tag{65}$$

It is very easy to show Eqs. (64) and (65) from these norm definitions and their relations. Using Eq. (49), we have

$$\|w\|^2_{\mathbb{G}^1(\Omega)} = \|w\|^2_{\mathbb{L}^2} + |w|^2_{\mathbb{G}^1(\Omega)}, \quad \forall w \in \mathbb{G}_h^1. \tag{66}$$

Since the L^2 norm is positive and the G^1 semi-norm is semi-positive for any nonzero $w \in \mathbb{G}_h^1$, we shall always have Eq. (65). When w is zero, all these norms becomes zero, and hence Eq. (64) holds.

3.4.3. *Scalar mortification*

Remark 3.3. A function in a $\mathbb{G}_h^1$ space is scalable in norm:

$$\|\alpha w\|_{\mathbb{G}^1} = |\alpha| \|w\|_{\mathbb{G}^1}, \quad \forall \alpha \in \mathbb{R}, \ \forall w \in \mathbb{G}_h^1, \tag{67}$$

which can be observed simply from the definition, say for example Eq. (46).

3.4.4. *Completeness*

Remark 3.4. A $\mathbb{G}_h^1$ is a complete metric (inner product induced norm) linear space, and is a Banach space [Peraire (1999)], meaning that for every Cauchy sequence w_j in $\mathbb{G}_h^1$ has a limit $v \in \mathbb{G}_h^1$.

A Cauchy sequence is one such that $\|w_j - w_k\|_{\mathbb{G}^1(\Omega)} \to 0$, $j, k \to \infty$, and the completeness requires that $\|v - w_j\|_{\mathbb{G}^1(\Omega)} \to 0$ as $j \to \infty$. The completeness property can be understood easily because a $\mathbb{G}_h^1$ space is a subspace of $\mathbb{L}^2$ space that is a Banach space [Susanne *et al.* (2008)]. Note also that the norm of $\mathbb{G}_h^1$ is a linear combination of $\mathbb{L}^2$ norms. Another way may be using the 1st inequality of equivalence of the G^1 norm of a function $w \in \mathbb{G}_h^1$ and the L^2 norm of $\mathbf{d} \in \mathbb{R}^{N_n}$ (see proof

given in Sec. 3.6), and the fact that $\mathbb{R}^{N_n}$ is known being a Banach space (that is complete).

3.4.5. *Cauchy-Schwarz inequality*

Remark 3.5. $\mathbb{G}_h^1$ inner product space: Cauchy-Schwarz inequality.

The $\mathbb{G}_h^1$ inner product defined in Eq. (46) is qualified bilinear form, and hence we shall have the Cauchy-Schwarz inequality:

$$|(\overline{w'}, \overline{v'})_{\mathbb{L}^2(\Omega)}| \leq |w|_{\mathbb{G}^1(\Omega)} \cdot |v|_{\mathbb{G}^1(\Omega)}, \tag{68}$$

and

$$|(w, v)_{\mathbb{G}^1(\Omega)}| \leq \|w\|_{\mathbb{G}^1(\Omega)} \cdot \|v\|_{\mathbb{G}^1(\Omega)}. \tag{69}$$

To show this, we first observe the symmetric, because swapping places for w and v will not change the value of the inner product. Second, it is positive definite, because of the positivity of the $(w, w)_{\mathbb{L}^2(\Omega)}$ and semi-positivity of $(\overline{w'}, \overline{w'})_{\mathbb{L}^2(\Omega)}$. Finally, it is bilinear, because of the bilinear property of $(w, v)_{\mathbb{L}^2(\Omega)}$and $(\overline{w'}, \overline{v'})_{\mathbb{L}^2(\Omega)}$. Equations (68) and (69) are fundamentally important for the functional analysis of G spaces.

3.4.6. *Triangular inequality*

We now prove the *triangular inequality* for $\mathbb{G}_h^1$ norm:

$$\|w + v\|_{\mathbb{G}^1} \leq \|w\|_{\mathbb{G}^1} + \|v\|_{\mathbb{G}^1}, \quad \forall w \in \mathbb{G}_h^1, \ \forall v \in \mathbb{G}_h^1. \tag{70}$$

We first proof this for 2D scalar functions:

$$\begin{aligned}
\|w + v\|_{\mathbb{G}^1} &= \left[\int_\Omega (w + v)^2 \mathrm{d}\Omega + \sum_{k=1}^{N_s} A_k^s |\overline{\nabla(w + v)}|^2\right]^{1/2} \\
&= \left[\begin{array}{l} \displaystyle\int_\Omega w^2 \mathrm{d}\Omega + \int_\Omega v^2 \mathrm{d}\Omega + 2\int_\Omega wv \mathrm{d}\Omega \\ \displaystyle\quad + \sum_{k=1}^{N_s} A_k^s ((\bar{g}_1(w) + \bar{g}_1(v))^2 + (\bar{g}_2(w) + \bar{g}_2(v))^2) \end{array}\right]^{1/2} \\
&= \left[\begin{array}{l} \displaystyle\int_\Omega w^2 \mathrm{d}\Omega + \int_\Omega v^2 \mathrm{d}\Omega + 2\int_\Omega wv \mathrm{d}\Omega \\ \displaystyle\quad + \sum_{k=1}^{N_s} A_k^s \left(\begin{array}{l} \bar{g}_1^2(w) + 2\bar{g}_1(w)\bar{g}_1(v) + \bar{g}_1^2(v) \\ \quad + \bar{g}_2^2(w) + 2\bar{g}_2(w)\bar{g}_2(v) + \bar{g}_2^2(v) \end{array}\right) \end{array}\right]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\begin{array}{l}
\underbrace{\int_\Omega w^2 \mathrm{d}\Omega + \sum_{k=1}^{N_s} A_k^s (\bar{g}_1^2(w) + \bar{g}_2^2(w))}_{\|w\|_{G^1}^2} \\
\quad + \underbrace{\int_\Omega v^2 \mathrm{d}\Omega + \sum_{k=1}^{N_s} A_k^s (\bar{g}_1^2(v) + \bar{g}_2^2(v))}_{\|v\|_{G^1}^2} \\
\quad + 2 \left(\underbrace{\int_\Omega wv \mathrm{d}\Omega + \sum_{k=1}^{N_s} A_k^s (\bar{g}_1(w)\bar{g}_1(v) + \bar{g}_2(w)\bar{g}_2(v))}_{(w,v)_{G^1} \le \|w\|_{G^1}\|v\|_{G^1}} \right)
\end{array} \right]^{1/2} \\
&\le \left[\|w\|_{G^1}^2 + 2\|w\|_{G^1}\|v\|_{G^1} + \|v\|_{G^1}^2 \right]^{1/2} \\
&= \|w\|_{G^1} + \|v\|_{G^1}, \quad \forall w \in \mathbb{G}_h^1, \ \forall v \in \mathbb{G}_h^1
\end{aligned} \tag{71}
$$

In the above proof process, we have used the Cauchy-Schwarz inequality for our inner product induced norms.

The exact same procedure can be applied to prove the triangular inequality for vector functions, but it will be a little lengthy. We show the process here only for the 2D case, by examining first the semi-norm of the sum of two functions $w, v \in \mathbb{G}^1$ based on the definition Eq. (56):

$$
\begin{aligned}
|w+v|_{\mathbb{G}^1(\Omega)}^2 &= \sum_{k=1}^{N_s} A_k^s \left(\begin{array}{l} \bar{g}_{11}^2(w_1+v_1) + \bar{g}_{22}^2(w_2+v_2) \\ \quad + (\bar{g}_{12}(w_1+v_1) + \bar{g}_{21}(w_2+v_2))^2 \end{array} \right) \\
&= \sum_{k=1}^{N_s} A_k^s \left(\begin{array}{l} \bar{g}_{11}^2(w_1) + \bar{g}_{11}^2(v_1) + 2\bar{g}_{11}(w_1)\bar{g}_{11}z(v_1) + \bar{g}_{22}^2(w_2) \\ \quad + \bar{g}_{22}^2(v_2) + 2\bar{g}_{22}(w_2)\bar{g}_{22}(v_2) + ((\bar{g}_{12}(w_1) + \bar{g}_{21}(w_2)) \\ \quad + (\bar{g}_{12}(v_1) + \bar{g}_{21}(v_2)))^2 \end{array} \right) \\
&= \left(\begin{array}{l} \underbrace{\sum_{k=1}^{N_s} A_k^s (\bar{g}_{11}^2(w_1) + \bar{g}_{22}^2(w_2) + (\bar{g}_{12}(w_1) + \bar{g}_{21}(w_2))^2)}_{|w|_{\mathbb{G}^1(\Omega)}^2} \\ \quad + \underbrace{\sum_{k=1}^{N_s} A_k^s (\bar{g}_{11}^2(v_1) + \bar{g}_{22}^2(v_2) + (\bar{g}_{12}(v_1) + \bar{g}_{21}(v_2))^2)}_{|v|_{\mathbb{G}^1(\Omega)}^2} \\ \quad + 2\sum_{k=1}^{N_s} A_k^s \left(\begin{array}{l} \bar{g}_{11}(w_1)\bar{g}_{11}(v_1) + \bar{g}_{22}(w_2)\bar{g}_{22}(v_2) \\ \quad + (\bar{g}_{12}(w_1) + \bar{g}_{21}(w_2))(\bar{g}_{12}(v_1) + \bar{g}_{21}(v_2)) \end{array} \right) \end{array} \right).
\end{aligned} \tag{72}
$$

We then examine the full norm of the sum of two functions $w, v \in \mathbb{G}_h^1$ based on the definition Eq. (57):

$$\begin{aligned}\|w+v\|^2_{\mathbb{G}^1(\Omega)} &= \int_\Omega ((w_1+v_1)^2+(w_2+v_2)^2)\mathrm{d}\Omega + |w+v|^2_{\mathbb{G}^1(\Omega)}\\ &= \int_\Omega (w_1^2+v_1^2+w_2^2+v_2^2+2w_1v_1+2w_2v_2)\mathrm{d}\Omega + |w+v|^2_{\mathbb{G}^1(\Omega)}\\ &= \int_\Omega (w_1^2+v_1^2)\mathrm{d}\Omega + \int_\Omega (w_2^2+v_2^2)\mathrm{d}\Omega\\ &\quad + 2\int_\Omega (w_1v_1+w_2v_2)\mathrm{d}\Omega + |w+v|^2_{\mathbb{G}^1(\Omega)}. \end{aligned} \tag{73}$$

Substituting Eq. (72) into (73) gives

$$\begin{aligned}\|w+v\|^2_{\mathbb{G}^1} &= \int_\Omega (w_1^2+v_1^2)\mathrm{d}\Omega + \int_\Omega (w_2^2+v_2^2)\mathrm{d}\Omega + 2\int_\Omega (w_1v_1+w_2v_2)\mathrm{d}\Omega\\ &\quad + |w|^2_{\mathbb{G}^1} + |v|^2_{\mathbb{G}^1} + 2\sum_{k=1}^{N_s} A_k^s \begin{pmatrix} +\bar{g}_{11}(w_1)\bar{g}_{11}(v_1)+\bar{g}_{22}(w_2)\bar{g}_{22}(v_2)\\ +(\bar{g}_{12}(w_1)+\bar{g}_{21}(w_2))(\bar{g}_{12}(v_1)\\ +\bar{g}_{21}(v_2))\end{pmatrix}\\ &= \|w\|^2_{\mathbb{G}^1} + \|v\|^2_{\mathbb{G}^1}\\ &\quad + 2\underbrace{\left(\begin{array}{l}\displaystyle\int_\Omega (w_1v_1+w_2v_2)\mathrm{d}\Omega\\ \displaystyle+\sum_{k=1}^{N_s} A_k^s \begin{pmatrix}\bar{g}_{11}(w_1)\bar{g}_{11}(v_1)+\bar{g}_{22}(w_2)\bar{g}_{22}(v_2)\\ +(\bar{g}_{12}(w_1)+\bar{g}_{21}(w_2))(\bar{g}_{12}(v_1)+\bar{g}_{21}(v_2))\end{pmatrix}\end{array}\right)}_{(w,v)\le \|w\|_{\mathbb{G}^1(\Omega)}\|v\|_{\mathbb{G}^1(\Omega)}},\\ &\le \|w\|^2_{\mathbb{G}^1} + \|v\|^2_{\mathbb{G}^1} + 2\|w\|_{\mathbb{G}^1}\|v\|_{\mathbb{G}^1}\\ &= (\|w\|_{\mathbb{G}^1} + \|v\|_{\mathbb{G}^1})^2 \end{aligned} \tag{74}$$

which is Eq. (70). Note here we used again the useful Cauchy-Schwarz inequality for inner product induced norms.

Comparing Eqs. (49) with (48), we obtain

$$|w|_{\mathbb{G}^1(\Omega)} \le \|w\|_{\mathbb{G}^1(\Omega)}, \quad \forall w \in \mathbb{G}_h^1, \tag{75}$$

meaning that the $\mathbb{G}_h^1$ full norm is always larger than the $\mathbb{G}_h^1$ semi-norm. This is obvious.

3.5. *Convergence property for functions in G^1 space*

Remark 3.6. *Convergence property*: For $w, v \in \mathbb{H}^1$, when $N_s \to \infty$ and all $\Omega_i^s \to 0$, $\bar{W}$ becomes Delta functions and the integral representation becomes exact. At such a limit, we have $\overline{\nabla w} \to \nabla w$, $(w, v)_{\mathbb{G}^1(\Omega)} \to (w, v)_{\mathbb{H}^1(\Omega)}$, $\|w\|_{\mathbb{G}^1(\Omega)} \to \|w\|_{\mathbb{H}^1(\Omega)}$, $|w|_{\mathbb{G}^1(\Omega)} \to |w|_{\mathbb{H}^1(\Omega)}$, $\|w\|_{\mathbb{G}^2(\Omega)} \to \|w\|_{\mathbb{H}^2(\Omega)}$, and $|w|_{\mathbb{G}^2(\Omega)} \to |w|_{\mathbb{H}^2(\Omega)}$.

Remark 3.6 ensures that all the bound properties for G^1 norms convergence to the corresponding H^1 norms defined in the same manner at the limit of $N_s \to \infty$ and all $\Omega_i^s \to 0$ for all functions in an H^1 space. We, however, need the inequalities for finite smoothing domains and for all functions in G^1 spaces, which are termed as G inequalities to be derived in the next sections.

3.6. *First inequality for functions in G^1 space*

The first inequality relates the (full) G^1 norm of a function to L^2 norm of the nodal values of the function when the function is approximated based on an approximation method using local nodes scattered in the problem domain. We state

Remark 3.7. 1st *inequality*: Functions in a $\mathbb{G}_h^1$ space, there exists a nonzero positive constant c_{wd}^f, such that the 1st inequality

$$\|\mathbf{d}\|_{\mathbb{L}^2(\Omega)} \geq c_{dw}^f \|w\|_{\mathbb{G}^1(\Omega)}, \quad \forall w \in \mathbb{G}_h^1, \tag{76}$$

or equivalently

$$\|w\|_{\mathbb{G}^1(\Omega)} \geq c_{wd}^f \|\mathbf{d}\|_{\mathbb{L}^2(\Omega)}, \quad \forall w \in \mathbb{G}_h^1, \tag{77}$$

holds. Equations (76) or (77) means that the full G^1 norm of a function in a $\mathbb{G}_h^1$ space is *equivalent* (different only with a factor that is independent of $w \in \mathbb{G}_h^1$) to the L^2 norm of the nodal values of the function.

Proof. The proof of Eq. (76) or (77) is straightforward. It uses the definition of the $\mathbb{G}_h^1$ space. Because any function w in $\mathbb{G}_h^1$ space is constructed using $w(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x})\mathbf{d}$, and because the nodal shape functions $\boldsymbol{\varphi}(\mathbf{x})$ are linearly independent, the full G^1 norm of w must be equivalent to the L^2 norm of the nodal values of the function $\mathbf{d}$. □

A more detailed analysis of Eqs. (76) or (77) for 1D cases and their extension to higher dimensions have been given in [Liu (2008b)].

3.7. *Second inequality*

The second inequality relates the G^1 semi-norm of a function to the L^2 norm of the nodal values of the function in a G^1 with a set of at least a minimum number of linearly independent smoothing domains [Liu (2008a)] created properly for evaluating the G^1 semi-norm. We now present the following remark.

Remark 3.8. 2nd inequality:

$$|w|_{\mathbb{G}^1(\Omega)} \geq c^s_{wd}\|\mathbf{d}\|_{\mathbb{L}^2(\Omega)}, \quad \forall w \in \mathbb{G}^1_{h,0}(\Omega), \tag{78}$$

or equivalently

$$\|\mathbf{d}\|_{\mathbb{L}^2(\Omega)} \geq c^s_{dw}|w|_{\mathbb{G}^1(\Omega)}, \quad \forall w \in \mathbb{G}^1_{h,0}(\Omega), \tag{79}$$

meaning that the full G^1 norm of a function in a $\mathbb{G}^1_h$ space is equivalent to the L^2 norm of the nodal values of the function.

Proof. The proof of the 2nd inequality is also straightforward. It uses again the definition of the $\mathbb{G}^1_h$ space. Because any function w in $\mathbb{G}^1_h$ space is constructed using $w(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x})\mathbf{d}$, the nodal shape functions $\boldsymbol{\varphi}(\mathbf{x})$ are linearly independent, and the positivity condition that $\sum_{k=1}^{N_s}(\int_{\Gamma^s_k} v(\mathcal{S})n_i \mathrm{d}\mathcal{S})^2 > 0$, for all $v \neq 0$, $\forall \mathbf{d} \in \mathbb{R}^{N_n}$ and $i = 1, \ldots, d$, the semi G^1 norm of w must be equivalent to the L^2 norm of the nodal values of the function $\mathbf{d}$. □

A more detailed analysis of Eqs. (78) or (79) for 1D case and their extension to higher dimensions can be found in using the concept of "positivity relay" [Liu (2008b)].

So far we have successfully developed node-based, edge-based, and cell-based smoothing domains, in both FEM and meshfree settings, as detailed in [Liu, Dai and Nguyen (2007); Liu and Nguyen *et al.* (2007); Dai, Liu and Nguyen (2007); Liu and Zhang *et al.* (2005); Zhang, Liu and Wang *et al.* (2007); Liu and Zhang (2008a, b); Zhang, Liu and Nguyen *et al.* (2007); Liu (2008a, b); Liu and Nguyen *et al.* (2009); Liu, Nguyen and Lam (2008); Wu *et al.* (2008); Nguyen, Liu and Lam *et al.* (2008); Liu and Zhang (2009)].

3.8. *Third inequality*

We now ready to present the 3rd inequality stated in the following theorem.

Theorem 3.1. *Equivalence of G norms*: Functions in a $\mathbb{G}^1_h$ space, there exists a positive nonzero constant c_G such that

$$c_G\|w\|_{\mathbb{G}^1(\Omega)} \leq |w|_{\mathbb{G}^1(\Omega)}, \quad \forall w \in \mathbb{G}^1_{h,0}, \tag{80}$$

meaning that the G^1 full norm and the G^1 semi-norm of any function in a $\mathbb{G}^1_{h,0}$ space are equivalent.

Proof. The combination of the inequality Eq. (76), and the inequality Eq. (78) gives

$$|w|_{\mathbb{G}^1(\Omega)} \geq c^s_{wd}\|d\|_{\mathbb{L}^2(\Omega)} \geq \underbrace{c^s_{wd}c^f_{dw}}_{c_G}\|w\|_{\mathbb{G}^1(\Omega)} \geq c_G\|w\|_{\mathbb{G}^1(\Omega)}, \quad \forall w \in \mathbb{G}^1_{h,0}(\Omega), \tag{81}$$

which is the third inequality. □

Combination of Eqs. (75) and (81), we arrived at the following chain inequalities.

$$c_G\|w\|_{\mathbb{G}^1(\Omega)} \leq |w|_{\mathbb{G}^1(\Omega)} \leq \|w\|_{\mathbb{G}^1(\Omega)}, \quad \forall w \in \mathbb{G}^1_{h,0}. \tag{82}$$

The 3rd inequality Eq. (80) is a generalized version of the well-known Poincare-Friedrichs inequality. It is the foundation of the W^2 formulation, ensuring the stability of the solution. Equation (82) is essential to ensure both the uniqueness and convergence of a W^2 formulation of a *physically* stable problem. For solid mechanic problems, for example, we need the material being stable [Liu (2008b)].

3.9. *Softening effects*

We further examine some of the important properties of functions in G spaces.

Remark 3.9. For a function in an H^1 space, the G^1 semi-norm of the function is no-larger than the H^1 semi-norm (of same type) of the function.

$$|w|_{\mathbb{G}^1(\Omega)} \leq |w|_{\mathbb{H}^1(\Omega)}, \quad \forall w \in \mathbb{H}^1, \tag{83}$$

meaning that the smoothing operation results in a smaller semi-norm measure. This is the fundamental inequality for the so-called softening effects. A proof on this remark can be found in [Liu (2008b)].

Because of Eq. (83), we immediately have

Remark 3.10. For a function in an H^1 space, the G^1 full norm of the function is no-larger than the H^1 full norm of the function:

$$\|w\|_{\mathbb{G}^1(\Omega)} \leq \|w\|_{\mathbb{H}^1(\Omega)}, \quad \forall w \in \mathbb{H}^1, \tag{84}$$

meaning that the smoothing operation results in a smaller (full) norm measure. This is because the first term in the RHS of Eq. (33) and that of Eq. (49) are exactly the same.

4. Conclusion

In this paper we have presented a general definition of G spaces, and provided an examination on the G space theory with an emphasis on the G^1 spaces. Our analyses and discussions on some of the key numerical treatments and these important inequalities for functions in G spaces led to the following major remarks:

(1) The generalized smoothing technique is very useful tool for evaluating the semi-norms of a function.
(2) The normed $\mathbb{G}^1_h$ space is linear space of finite dimension with inner product induced norms, and has all these basic properties of a linear space. It is a Banach space and hence is complete.

Table 1. Some important inequalities and properties for a $\mathbb{G}_h^1$ space.

Items	Properties	Expressions	Remarks
1.	1st inequality	$\|\mathbf{d}\|_{\mathbf{L}^2(\Omega)} \geq c_{dw}^f \ \|w\|_{\mathbb{G}^1(\Omega)}, \forall w \in \mathbb{G}_h^1$	Eq. (76)
		$\|w\|_{\mathbb{G}^1(\Omega)} \geq c_{wd}^f \|\mathbf{d}\|_{\mathbf{L}^2(\Omega)}, \forall w \in \mathbb{G}_h^1$	Eq. (77)
2.	2nd inequality	$\|w\|_{\mathbb{G}^1(\Omega)} \geq c_{wd}^s \|\mathbf{d}\|_{\mathbf{L}^2(\Omega)}, \forall w \in \mathbb{G}_{h,0}^1(\Omega)$	Eq. (78)
		$\|\mathbf{d}\|_{\mathbf{L}^2(\Omega)} \geq c_{dw}^s \|w\|_{\mathbb{G}^1(\Omega)}, \forall w \in \mathbb{G}_{h,0}^1(\Omega)$	Eq. (79)
3.	3rd inequality	$c_G \|w\|_{\mathbb{G}^1(\Omega)} \leq \|w\|_{\mathbb{G}^1(\Omega)}, \forall w \in \mathbb{G}_{h,0}^1$	Eq. (80)
4.	Ellipticity	$c_G \|w\|_{\mathbb{G}^1(\Omega)} \leq \|w\|_{\mathbb{G}^1(\Omega)} \leq \|w\|_{\mathbb{G}^1(\Omega)}, \forall w \in \mathbb{G}_{h,0}^1$	Eq. (82)
5.	Softening effect	$\|w\|_{\mathbb{G}^1(\Omega)} \leq \|w\|_{\mathbb{H}^1(\Omega)}, \forall w \in \mathbb{H}^1$	Eq. (84)
6.	Relations with others	$\mathbb{H}_h^1 \in \mathbb{G}_h^1 \in \mathbb{L}^2$	

(3) A normed $\mathbb{G}_h^1$ space can be created using a set of linearly independent nodal shape functions created properly for a set of nodes/cells for a given problem domain, and with as set of at least minimum number of independent smoothing domains created in a proper manner in relation to these nodes/cells.
(4) The key inequalities and properties for a $\mathbb{G}_h^1$ space are summarized in Table 1.
(5) Based on the $\mathbb{G}_h^1$ space theory, W^2 formulations of any physically stable problem will be stable and converge. It is applicable to any problems to which the standard weak formulation is applicable and will offer many attractive features.

Finally, we note that the G space theory is very new, and there are still many issues to be studied in a much greater detail, such as what would be properties of G^m (where m is non-negative integer larger than (1), what would be the dual space G^{-1}, what kind of linear functionals can we allow, etc. The author (as an engineer) hopes that our primary studies can initiate a more in-depth study in the G spaces and W^2 formulation related areas leading a class of even more effective computational methods.

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References

Chen, J. S., Wu, C. T. and Belytschko, T. [2000] Regularization of material instabilities by meshfree approximations with intrinsic length scales, *International Journal for Numerical Methods in Engineering* **47**, 1303–1322.

Chen, J. S., Wu, C. T., Yoon, S. and You, Y. [2001] A stabilized conforming nodal integration for Galerkin mesh-free methods, *International Journal for Numerical Methods in Engineering* **50**, 435–466.

Dai, K. Y., Liu, G. R. and Nguyen, T. T. [2007] An n-sided polygonal smoothed finite element method (nSFEM) for solid mechanics, *Finite Elements in Analysis and Design* **43**, 847–860.

Eringen, A. C. and Edelen, D. G. B. [1972] On nonlocal elasticity, *Int. J. Eng. Sci.* **10**, 233–248.

Hughes, T. J. R. [1987] *The Finite Element Method*, Prentice-Hall, London.

Kee, B. B. T, Liu, G. R. and Lu, C. [2007] A regularized least-squares radial point collocation method (RLS-RPCM) for adaptive analysis, *Computational Mechanics* **40** 837–853.

Liu, G. R. and Gu, Y. T. [2001] A point interpolation method for two-dimensional solids, *International Journal for Numerical Methods in Engineering* **50**, 937–951.

Liu, G. R. and Liu, M. B. [2003] *Smoothed Particle Hydrodynamics — A Meshfree Particle Method*. (World Scientific: Singapore).

Liu, G. R. and Quek, S. S. [2003] *Finite Element Method: A Practical Course* (Butterworth-Heinemann, Burlington, MA).

Liu, G. R. and Gu, Y. T. [2005] *An Introduction to MFree Methods and their Programming*, (Springer).

Liu, G. R., Zhang, G. Y., Dai, K. Y., Wang, Y. Y., Zhong, Z. H., Li, G.Y. and Han, X. [2005] A linearly conforming point interpolation method (LC-PIM) for 2D mechanics problems, *International Journal for Computational Methods* **2**, 645–665.

Liu, G. R. and Kee, B. B. T. [2006] A stabilized least-squares radial point collocation method (LS-RPCM) for adaptive analysis, *Computer Method in Applied Mechanics and Engineering* **195**, 4843–4861.

Liu, G. R., Li, Y., Dai, K. Y., Luan, M. T. and Xue, W. [2006] A linearly conforming radial point interpolation method for solid mechanics problems, *International Journal of Computational Methods* **3**, 401–428.

Liu, G. R., Dai, K. Y. and Nguyen, T. T. [2007] A smoothed finite element method for mechanics problems, *Computational Mechanics* **39**, 859–877.

Liu, G. R., Nguyen, T. T., Dai, K. Y. and Lam, K. Y. [2007] Theoretical aspects of the smoothed finite element method (SFEM), *International Journal for Numerical Methods in Engineering* **71**, 902–930.

Liu, G. R. [2008a] A generalized gradient smoothing technique and the smoothed bilinear form for galerkin formulation of a wide class of computational methods, *International Journal of Computational Methods* **5**(2), 199–236.

Liu, G. R. [2008b] A G space and weakened weak (W^2) form for a unified formulation of compatible and incompatible methods, part I–Theory and part II–Application to solid mechanics problems, *International Journal for Numerical Methods in Engineering*, accepted.

Liu, G. R. and Xu, G. X. [2008] A gradient smoothing method (GSM) for fluid dynamics problems, *Int. J. Numer. Methods Fluids* **56**(10), 1101–1133.

Liu, G. R. and Zhang, G. Y. [2008a] Edge-based smoothed point interpolation method (ES-PIM),*International Journal of Computational Methods* **5**(4), 621–646.

Liu, G. R. and Zhang, G. Y. [2008b] Upper bound solution to elasticity problems: A unique property of the linearly conforming point interpolation method (LC-PIM), *Int. J. Numer. Mech. Engrg.* **74**, 1128–1161.

Liu, G. R., Nguyen, T. T. and Lam, K. Y. [2008] An edge-based smoothed finite element method (E-SFEM) for static and dynamic problems of solid mechanics, *J. Sound. Vib.,* (in press), Doi:10.1016/j.jsv.2008.08.027.

Liu, G. R. [2009] *Meshfree Methods: Moving Beyond the Finite Element Method*, 2nd Edition (CRC Press, Boca Raton, USA).

Liu, G. R. and Zhang, G. Y. [2009] A normed G space and weakened weak (W^2) formulation of a cell-based smoothed point interpolation methods (CS-PIM), *International Journal of Computational Methods* **6**(1), 147–179.

Liu, G. R., Nguyen, T. T., Nguyen, X. H. and Lam, K. Y. [2009] A node-based smoothed finite element method (N-SFEM) for upper bound solutions to solid mechanics problems, *Computers and Structures* **87**, 14–26.

Liu, M. B., Liu, G. R. and Zong, Z. [2008] An overview on smoothed particle hydrodynamics, *International Journal of Computational Methods* **5**(1), 135–188.

Lucy, L. B. [1977] Numerical approach to testing the fission hypothesis, *Astronomical Journal* **82**, 1013–1024.

Monaghan, J. J. [1982] Why particle methods work (hydrodynamics), *SIAM Journal on Scientific and Statistical Computing* **3**, 422–433.

Nguyen, T. T., Liu, G. R., Lam, K. Y. and Zhang, G. Y. [2008] A face-based smoothed finite element method (F-SFEM) for 3D linear and nonlinear solid mechanics problems using 4-node tetrahedral elements, *International Journal for Numerical Methods in Engineering*, (in press) doi:10.1002/nme.2491.

Oliveira Eduardo, R. and De Arantes, E. [1968] Theoretical foundations of the finite element method. *International Journal of Solids and Structures* **4**, 929–952.

Peraire, J. [1999] *Lecture Notes on Finite Element Methods for Elliptic Problems*, (MIT).

Pian, T. H. H. and Wu, C. C. [2006] *Hybrid and Incompatible Finite Element Methods* (CRC Press, Boca Raton).

Strang, G. and Fix, G. J. [1973] *An Analysis of the Finite Element Method* (Prentice-Hall).

Susanne, C., Brenner, S. C. and Scott, L. R. [2008] *The Mathematical Theory of Finite Element Methods*, 3rd Edition, Springer.

Wang, J. G. and Liu, G. R. [2002] A point interpolation meshless method based on the radial basis functions, *International Journal for Numerical Methods in Engineering* **54**, 1623–1648.

Wikipedia-the Free Encyclopedia, http://en.wikipedia.org/wiki/Integration_by_parts.

Wu, S. C., Liu, G. R., Zhang, H. O. and Zhang, G. Y. [2008] A node-based smoothed point interpolation method (NS-PIM) for three-dimensional thermoelastic problems, *Numerical Heat Transfer, Part A: Applications* **54**, 1121–1147.

Xu, G. X., Liu, G. R. and Tani, A. [2009] An adaptive gradient smoothing method (GSM) for fluid dynamics problems, *Int. J. Numer. Methods Fluids*, Published Online, DOI: 10.1002/fld.2032.

Zhang, G. Y., Liu, G. R., Wang, Y. Y., Huang, H. T., Zhong, Z. H., Li, G. Y. and Han, X. [2007] A linearly conforming point interpolation method (LC-PIM) for three-dimensional elasticity problems, *International Journal for Numerical Methods in Engineering* **72**, 1524–1543.

Zhang, G. Y., Liu, G. R., Nguyen, T. T., Song, C. X., Han, X., Zhong, Z. H. and Li, G. Y. [2007] The upper bound property for solid mechanics of the linearly conforming radial point interpolation method (LC-RPIM), *International Journal of Computational Methods* **4**(3), 521–541.

Zhang, Y. Q., Liu, G. R. and Han, X. [2006] Effect of small length scale on elastic buckling of multi-walled carbon nanotubes under radial pressure, *Physics Letters A* **349**, 370–376.

Zienkiewicz, O. C. and Taylor, R. L. [2000] *The Finite Element Method* (5th edition), (Butterworth Heinemann, Oxford).